Phase-field fracture: past successes, current issues

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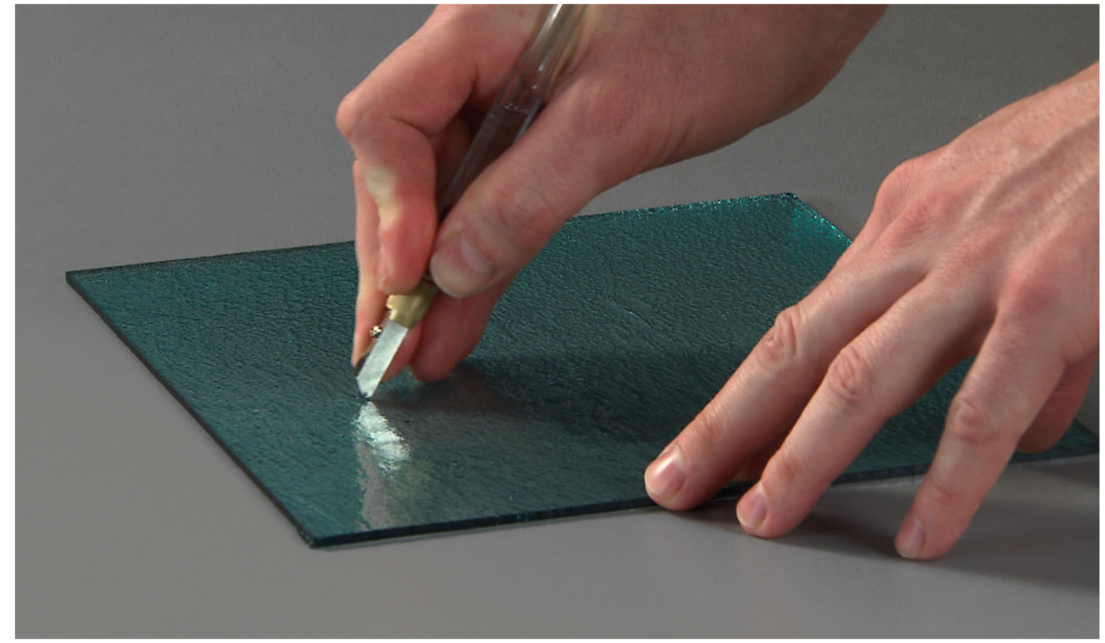
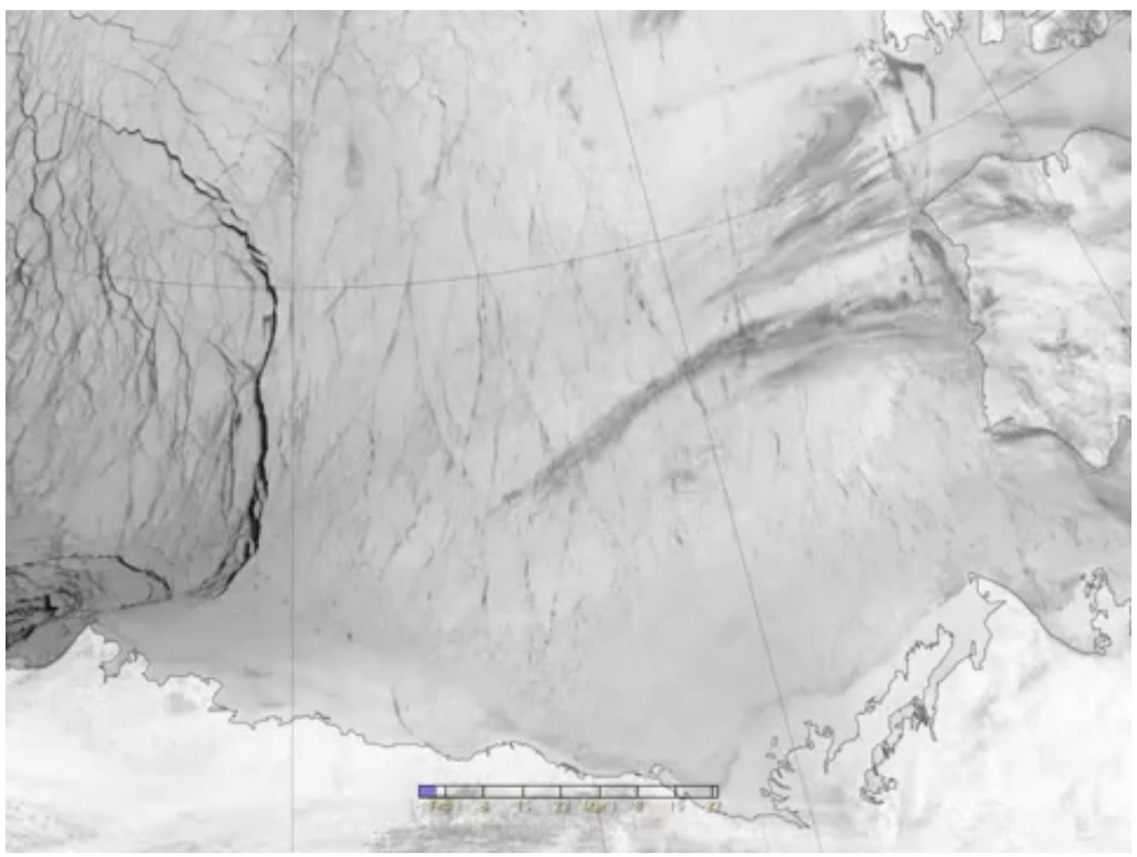
Fracture Mechanics

FIU pedestrian bridge, 2018



Glass “cutting”



Beaufort sea, 2013 (NASA earth observer)

Oil Painting (Danish Royal Academy)

Francfort and Marigo's Variational Approach to Fracture

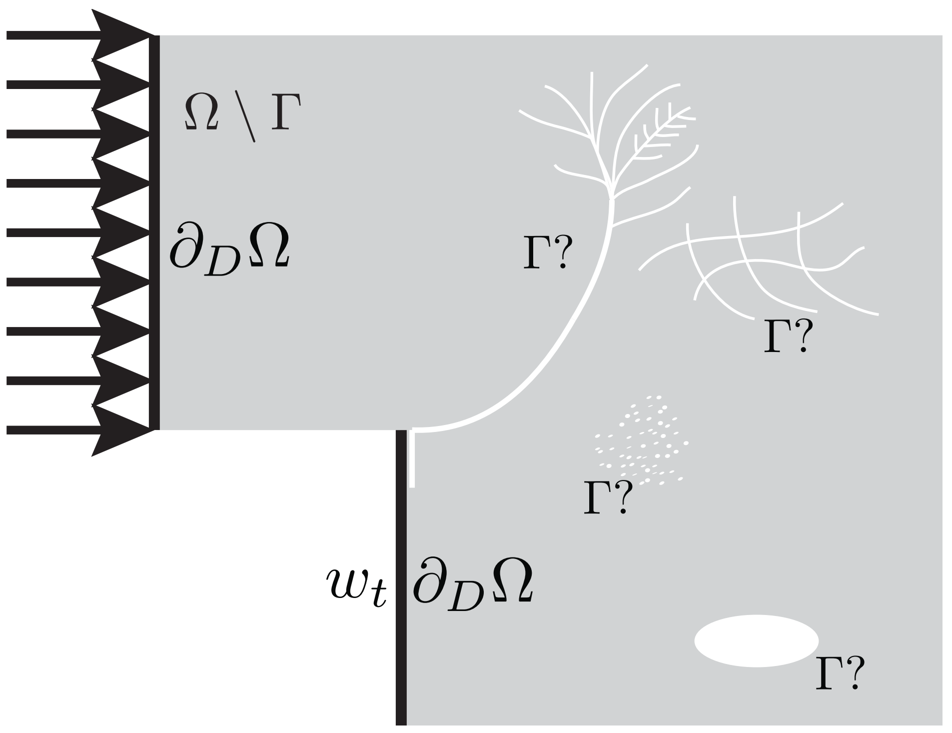
Modern view of Griffith's theory:

Displacement field u and crack set Γ given as unilateral minimizers of a free-discontinuity energy:

$$\mathscr{E}(u,\Gamma) := \int_{\Omega\setminus\Gamma} W(\mathrm{e}(u))\,dx + G_c\mathscr{H}^{n-1}(\Gamma)$$

amongst *all* admissible displacements fields $u(t)$ and *all* crack sets $\Gamma(t) \nearrow t$.

- $\mathrm{e}(u)$: linearized strain,
- $W\left(\mathrm{e}(u)\right) := \frac{1}{2}\mathbf{A}\mathrm{e}(u)\cdot\mathrm{e}(u)$: strain energy density,
- G_c: fracture toughness,
- $\mathscr{H}^{n-1}$: $n-1$–dimensional Hausdorff measure.



Variational phase-field approximation

Francfort and Marigo's variational view of Griffith's criterion:

$$\mathcal{E}(u,\Gamma) := \int_{\Omega\setminus\Gamma} W(\mathrm{e}(u))\,dx + G_c\mathcal{H}^{n-1}(\Gamma),\ W(\mathrm{e}(u)) := \frac{1}{2}\mathrm{Ae}(u)\cdot\mathrm{e}(u)$$

Phase-field approximation: $\ell > 0, 0 \leq \alpha \leq 1$:

$$\mathcal{E}_\ell(u,\alpha) := \int_\Omega a(\alpha)W(\mathrm{e}(u))\,dx + \frac{G_c}{4c_w}\int_\Omega \frac{w(\alpha)}{\ell} + \ell\,|\nabla\alpha|^2\,dx|$$

$$a(0) = 1,\ a(1) = 0,\ w(0) = 0,\ w(1) = 1, c_w = \int_0^1 \sqrt{w(s)}\,ds$$

Unilateral global minimization:

$$(u_i,\alpha_i) = \arg\min_{v,\alpha_{i-1}\leq\beta\leq 1} \mathcal{E}_\ell(v,\beta)$$

Γ-convergence of $\mathcal{E}_\ell$ to $\mathcal{E}$ + compactness of $\mathcal{E}_\ell \Longrightarrow$ convergence of minimizers.

AT$_1$: $$\mathcal{E}_\ell(u,\alpha) := \int_\Omega (1-\alpha)^2 W(\mathrm{e}(u))\,dx + \frac{3G_c}{8}\int_\Omega \frac{\alpha}{\ell} + \ell\,|\nabla\alpha|^2\,dx$$

Numerical implementation: `mef90/vDef`

Fortran90-2008, unstructured 2D/3D parallel finite elements.

- PETSC solvers, mesh management, I/O.
- Many variants of AT models, unilateral contact models.
- Perfect plasticity coupled with damage / fracture.
- Steady state / transient heat transfer coupled (one way) to fracture.

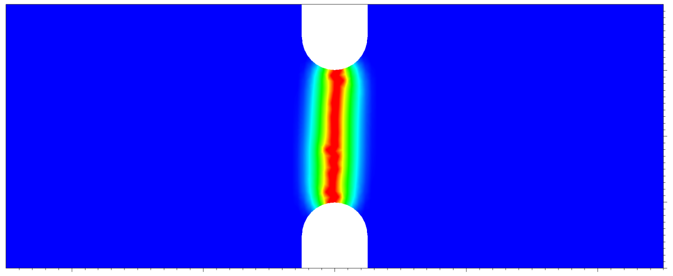
Main solver: time discrete alternate minimization (block Gauss-Seidel).

- Globally stable, *monotonically decreasing energy,* convergence to a critical point.

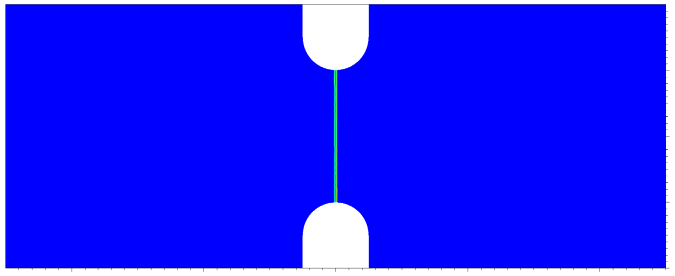
Other solvers: semi implicit gradient flows, quasi-Newton solvers, backtracking algorithm (optimality conditions in trajectory space).

Open source (BSD license) since 2014:

DOI:10.5281/zenodo.4290835
https://github.com/bourdin/mef90
dockerhub: bourdin/mef90ubuntumpicho

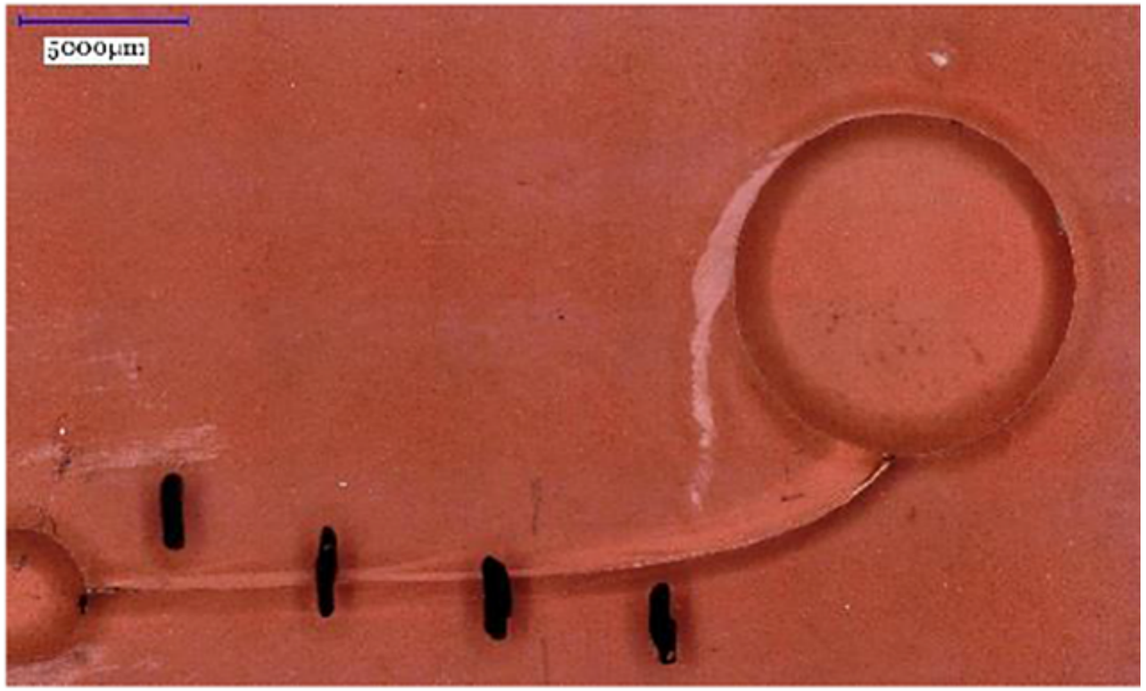


$\ell = 0.15$

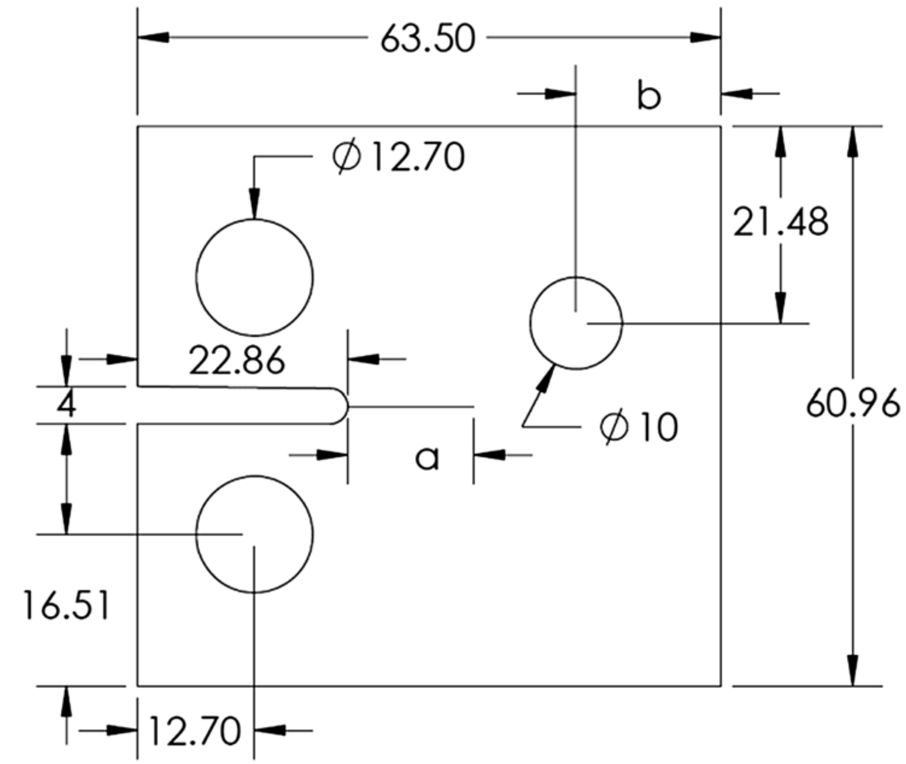


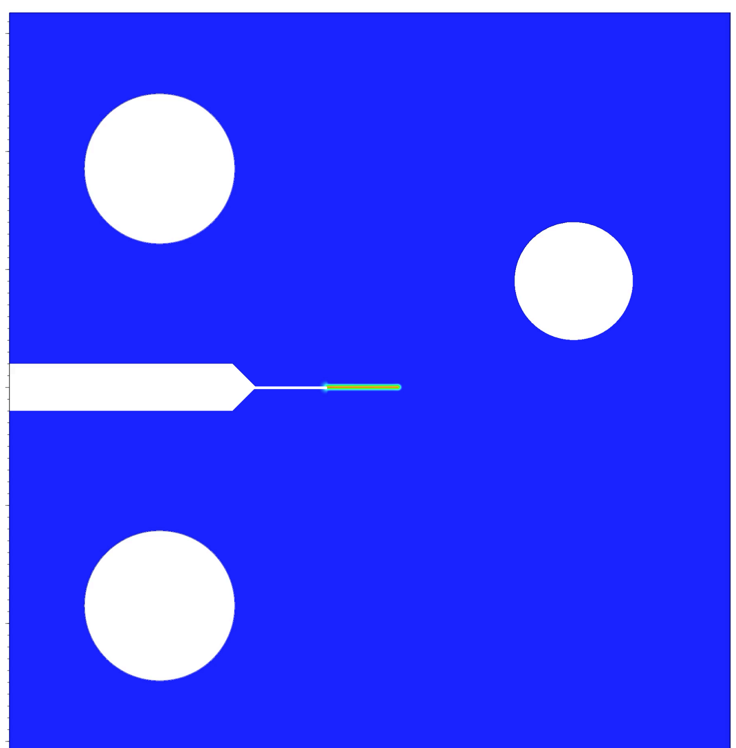
$\ell = 0.0075$

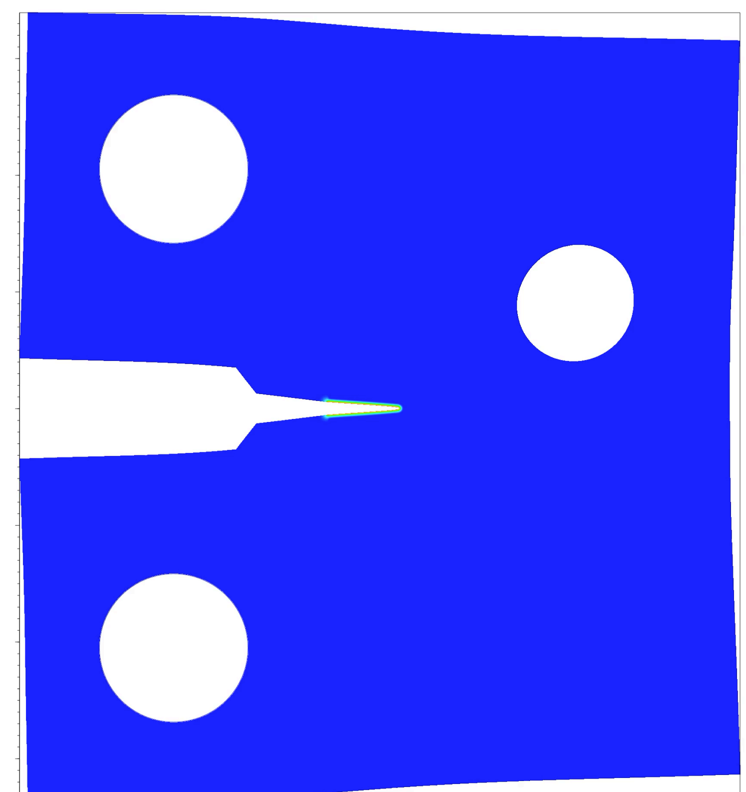
Variational Phase-Field fracture



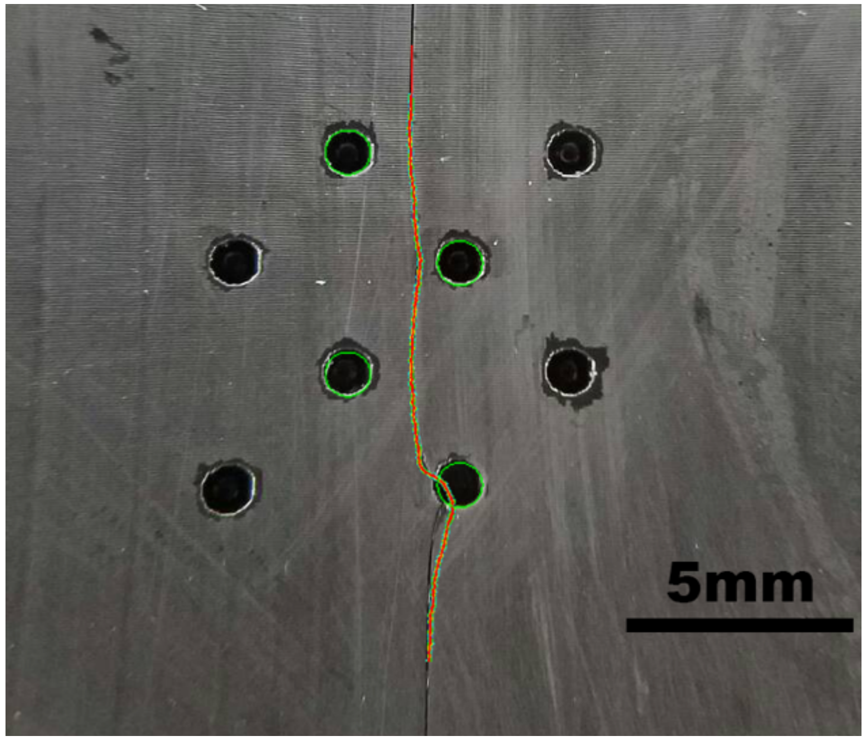
Pham Ravi-Chandar *IJF*, 2016



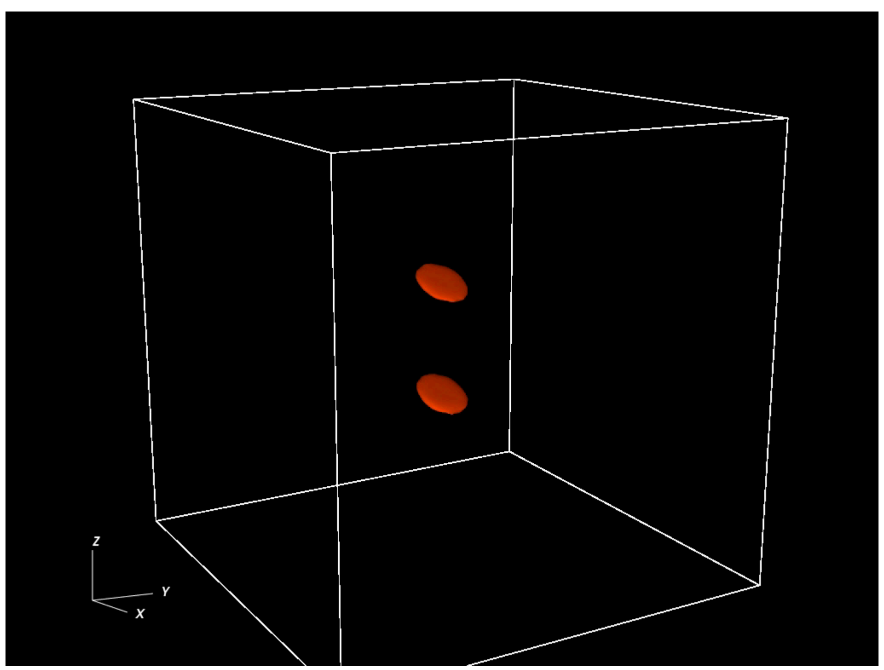




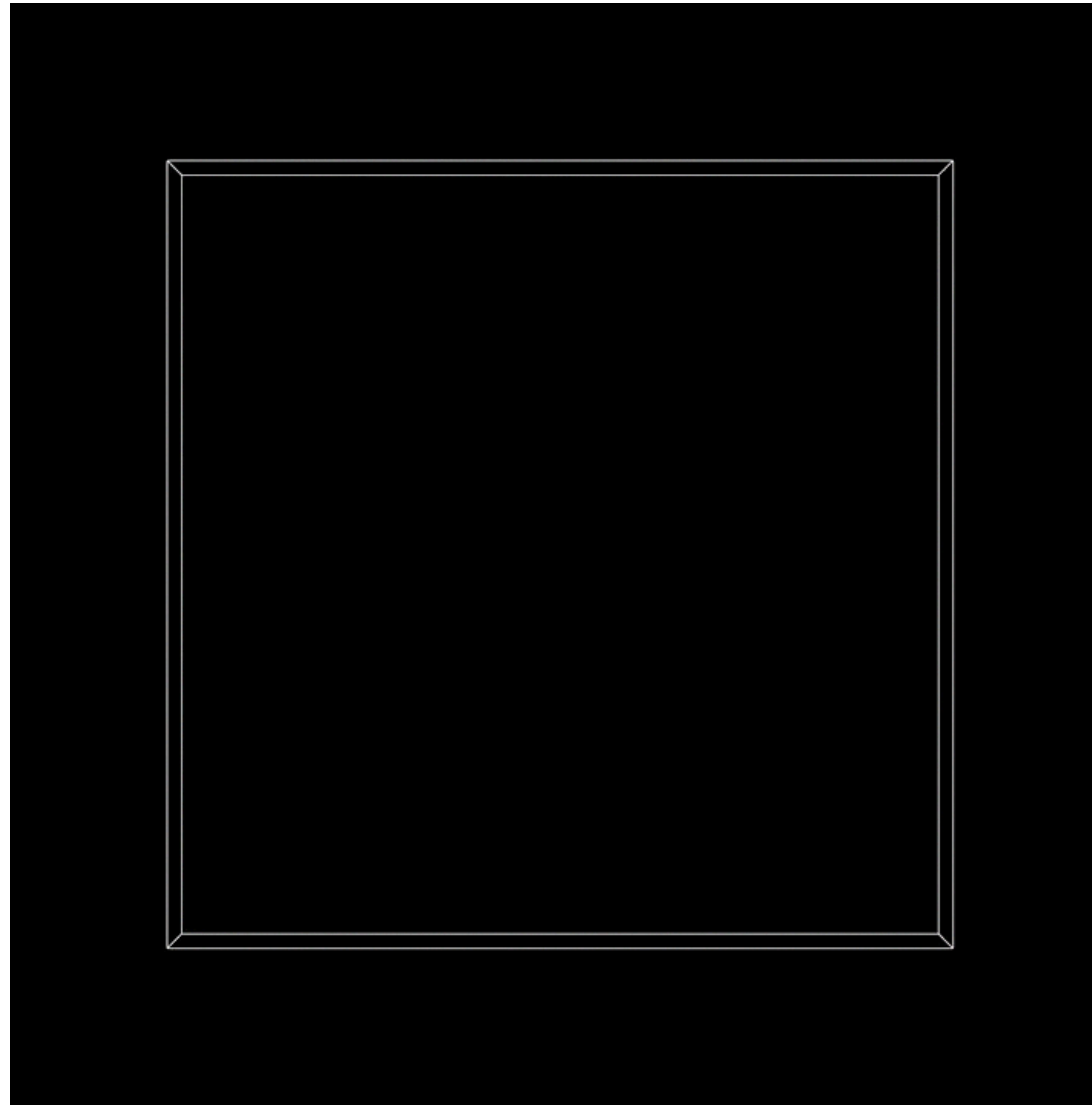
Variational Phase-Field fracture



Brodnik etAl *JAM* ’20



B Chukwudozie Yoshioka *SPE ATCE* ’12

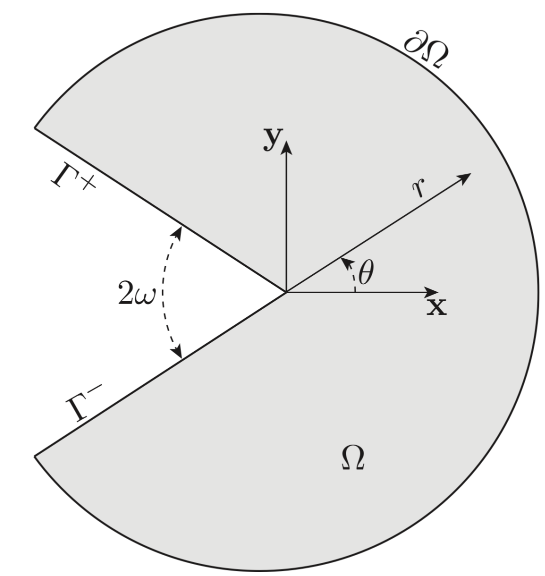


B-Maurini-Marigo-Sicsic, *PRL*, ‘14

Nucleation and Scale effects

Strength vs. toughness in Griffith theory

Crack *nucleation* is governed by *strength*,
propagation is governed by *toughness*.
Griffith's formalism cannot account for both.



Singularity near a re-entrant corner in mode-I:

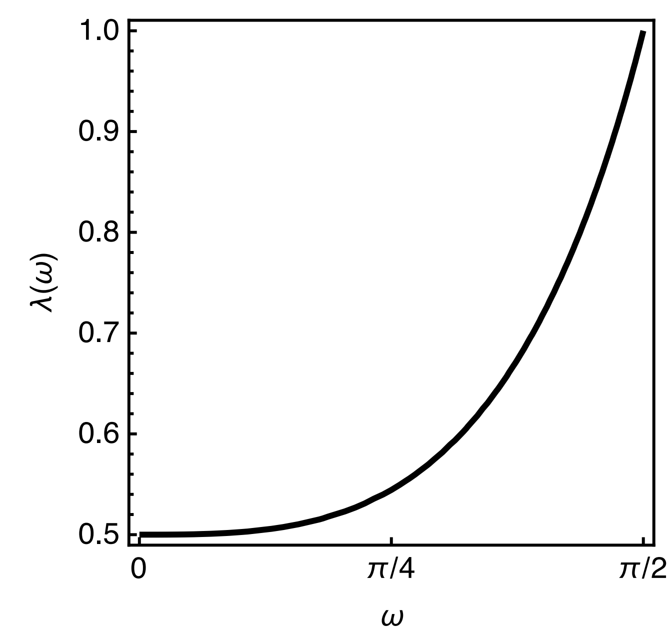
- $u(r,\theta) = \sigma_\infty \mathcal{O}\left(r^{\lambda(\omega)}\right)$
- $\sigma_{\theta\theta}(r,\theta=0) = \sigma_\infty \mathcal{O}\left(r^{\lambda(\omega)-1}\right)$
- $\mathcal{E}(\rho) = \sigma_\infty^2 \mathcal{O}\left(\rho^{2\lambda(\omega)}\right)$

Stability of a *infinitesimal* crack increment:

- Nucleation *only* possible if $\lambda(\omega) = 1/2$ $(\omega = 0)$.

Strength-based nucleation criterion:

- Nucleation for *any* load $\sigma_\infty > 0$ unless $\omega < \pi/2$.
- No localization if $\omega = \pi/2$ (no corner).



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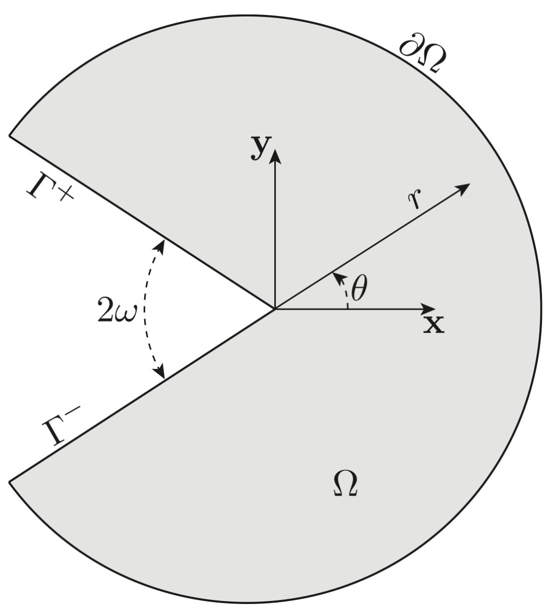
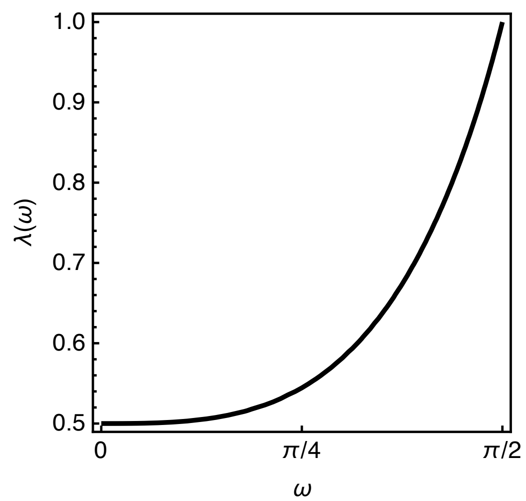
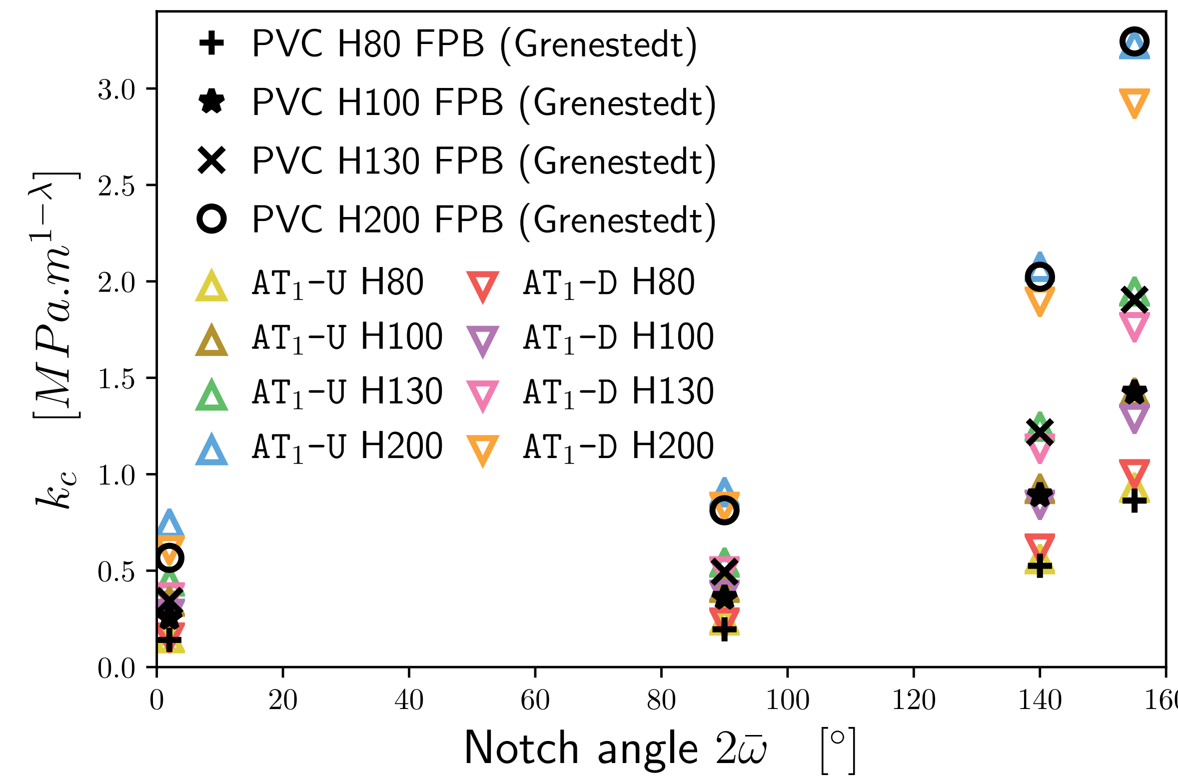
Nucleation in AT_1 (Tanné et al *JMPS*, 2018)

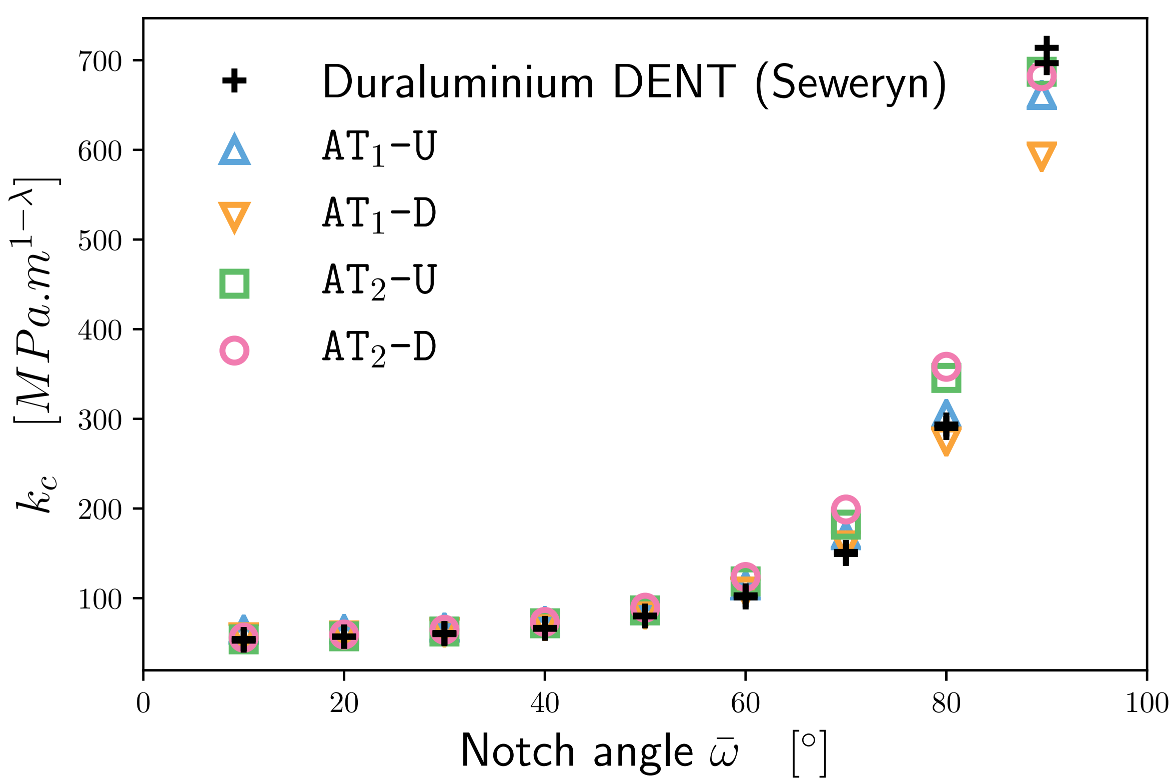
Nucleation at a V-notch

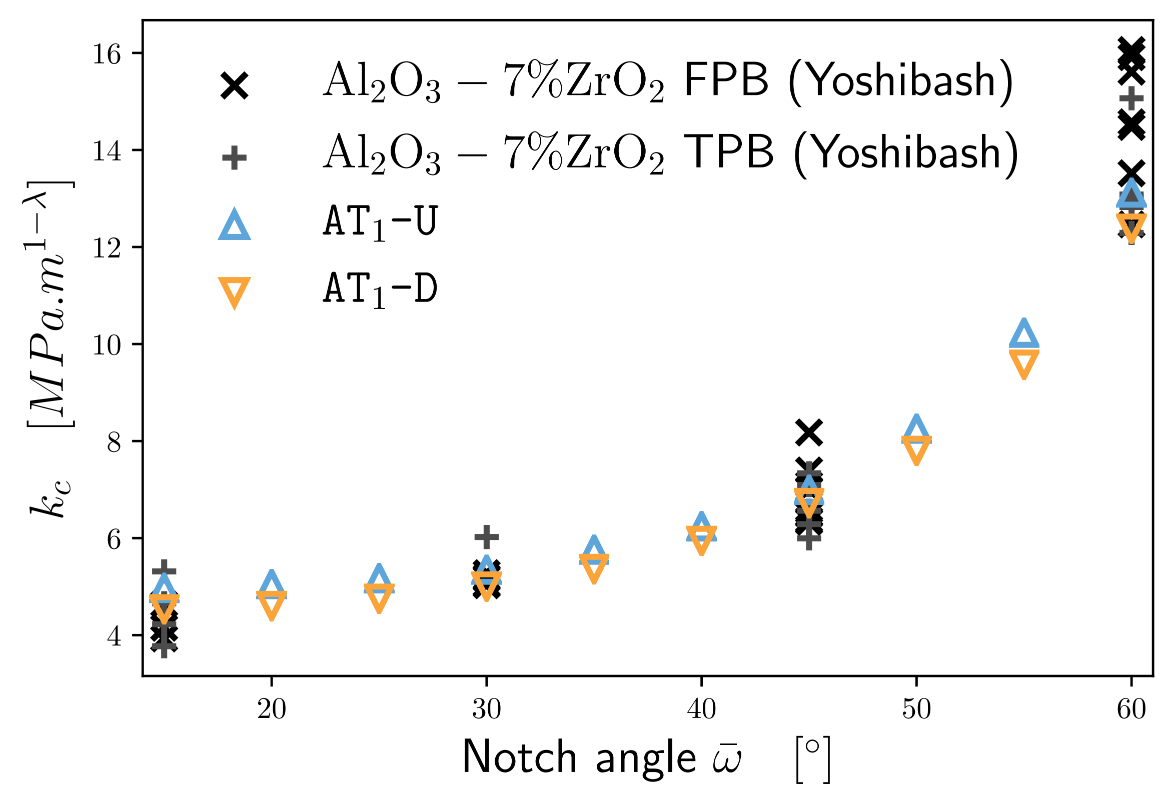




$$k := \lim_{r \to 0} \frac{\sigma_{\theta\theta}(r,0)}{(2\pi\, r)^{\lambda - 1}}$$

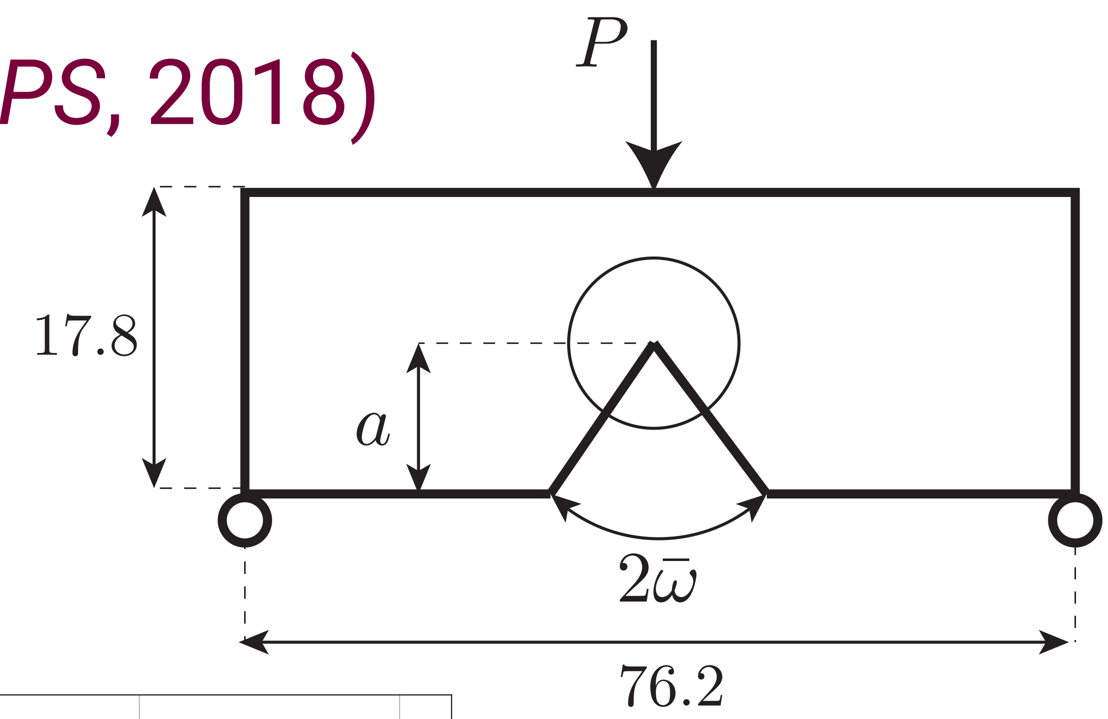


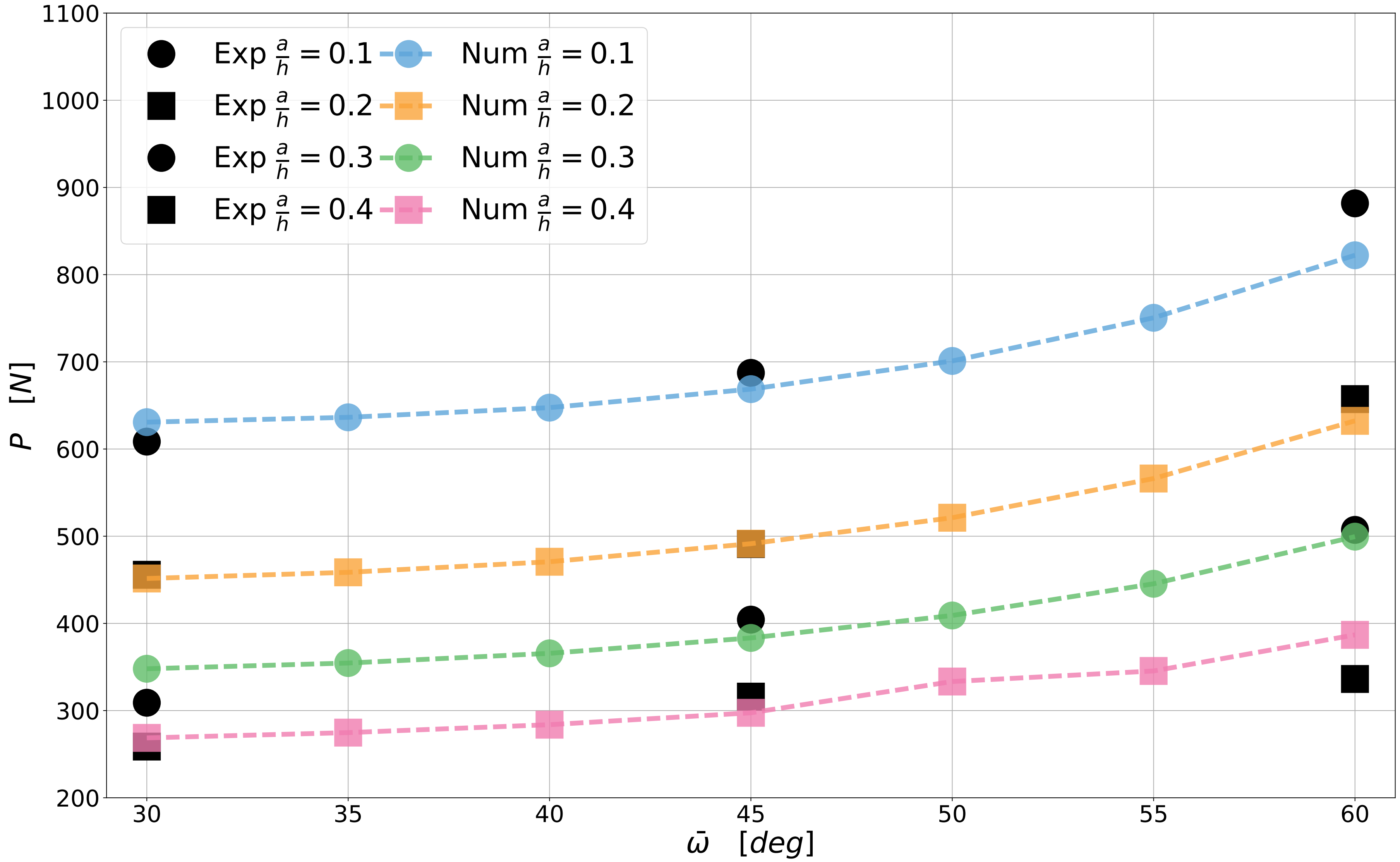




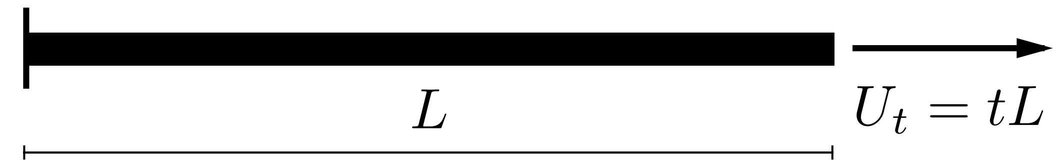
Nucleation in AT_1 (Tanné et al *JMPS*, 2018)

Nucleation at a V-notch





Nucleation in AT_1 (1D)



AT1 energy in 1D:

$$\mathcal{E}_\ell(u,\alpha) := \frac{1}{2}\int_0^L (1-\alpha)^2 E(u')^2\,dx + \frac{3G_c}{8}\int_0^L \frac{\alpha}{\ell} + \ell(\alpha')\,|^2\,dx$$

First order necessary conditions for optimality:

With respect to u:

$$\left[(1-\alpha)^2 E u'\right]' = 0.$$

With respect to α:

$$-(1-\alpha)E(u')^2 + \frac{3G_c}{8}\left(\frac{1}{\ell} - 2\ell\alpha''\right) \begin{cases} \geq 0 \text{ if } \alpha = \alpha_{i-1} \\ = 0 \text{ if } \alpha_{i-1} < \alpha < 1 \\ \leq 0 \text{ if } \alpha = 1 \end{cases}$$

Nucleation in AT_1 (1D)

Solutions of the NCO (cf. Pham et al *JMPS* 2011, *Meccanica* 2016, …):

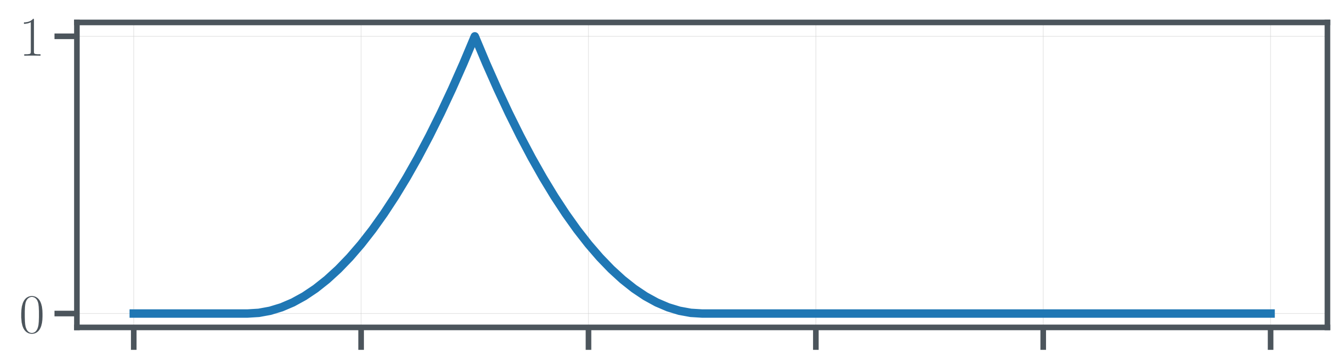
Elastic branch: $u_t(x) = tx,\ \alpha_t(t,x) = 0$, only if $t \leq t_e := \sqrt{\dfrac{3G_c}{8E\ell}}$.

Homogeneous damage: $u_t(x) = tx,\ \alpha_t(x) = 1 - \dfrac{3G_c}{8\ell E t^2}$, only if $t \geq t_e$.

Partially localized: $\alpha_t(x)$ smooth, non-constant, $\max_x \alpha_t(x) > 0$.

Fully localized: $u_t(x)$ *piecewise constant,* $\alpha_t(x)$ *optimal profile for* AT_1*:*

$$\alpha_t(x) = \begin{cases} \left(\dfrac{|x-x_0|}{2\ell} - 1\right)^2 & \text{if } |x-x_0| \leq 2\ell, \\ 0 & \text{otherwise.} \end{cases}$$



Nucleation in AT_1 (1D)

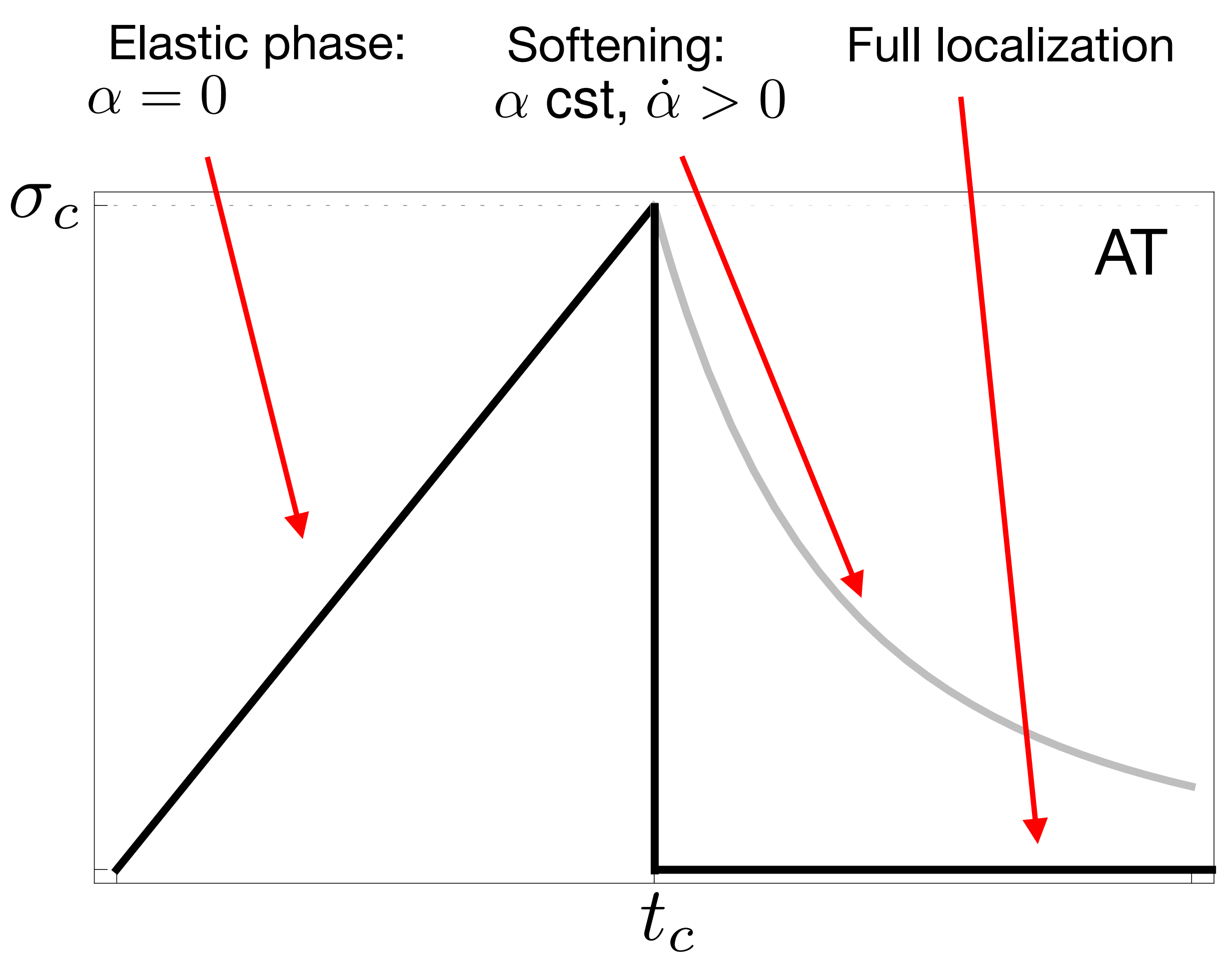
Stability analysis: (cf. Pham et al *JMPS* 2011, *Meccanica* 2016, …):

Elastic branch is *stable* if $t \leq t_c = t_e := \sqrt{\dfrac{3G_c}{8E\ell}}, \sigma_c = \sigma_e = \sqrt{\dfrac{3G_c E}{8\ell}}$.

Homogeneous damage, partially localized branch are *unstable*.

Fully localized branch is *stable*.

Link internal length and tensile strength: $\ell = \dfrac{3}{8}\dfrac{G_c E}{\sigma_c^2} = \dfrac{3}{8}\dfrac{K_{I,c}^2}{\sigma_c^2}$



Nucleation in AT_1 (Tanné et al *JMPS*, 2018)

Stress or energy criterion?

First order necessary conditions for optimality:

$$-\nabla \cdot \left[(1-\alpha)^2 \mathrm{Ae}(u)\right] = 0 + \text{BC}.$$

$$\left\{ -(1-\alpha)W(\mathrm{e}(u)) + \frac{3G_c}{8}\left(\frac{1}{\ell} - 2\ell\Delta\alpha\right) \quad \begin{array}{l} \geq 0 \text{ if } \alpha = \alpha_{i-1} \\ = 0 \text{ if } \alpha_{i-1} < \alpha < 1 \\ \leq 0 \text{ if } \alpha = 1 \end{array} \right.$$

Elastic state possible if $W(\mathrm{e}(u)) \leq \dfrac{3G_c}{8\ell}$, homogeneous states are unstable.

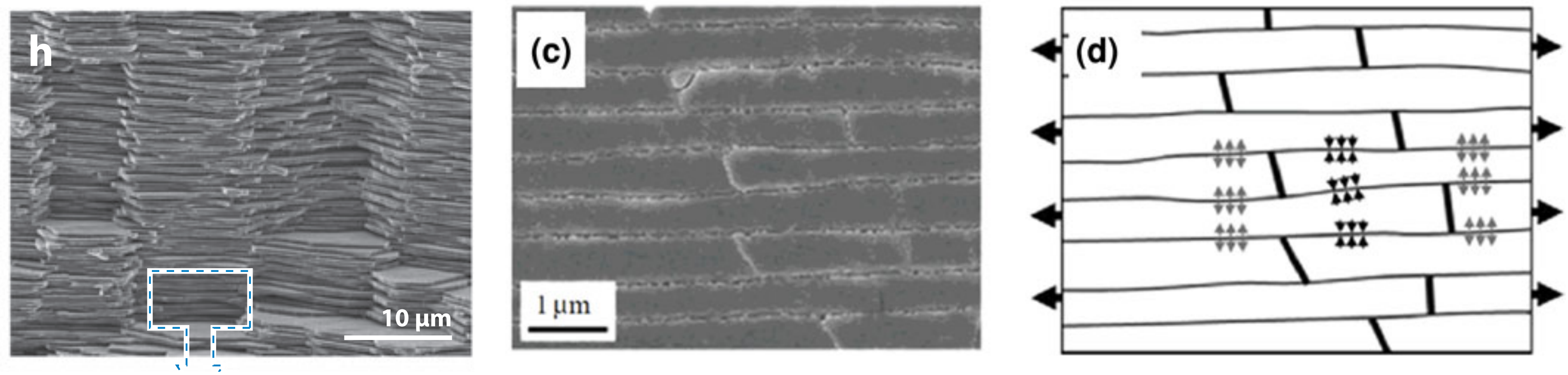
No construction of localized solutions (other than 1D).

Analysis of general case is lacking.

Loss of link with fracture, theoretical framework for evolution, uniqueness.

Fracture of Heterogeneous Materials

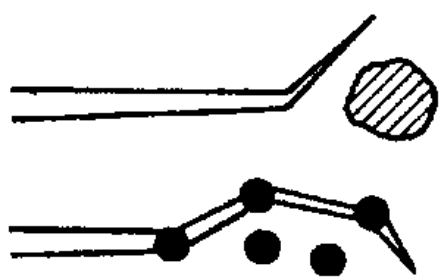
Fracture in heterogeneous materials



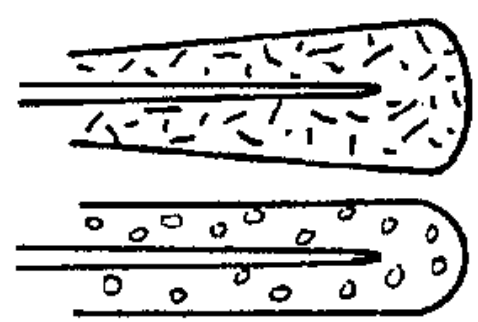
Yang Gupta '11
Salinas Kisalius, '13.

Goals:

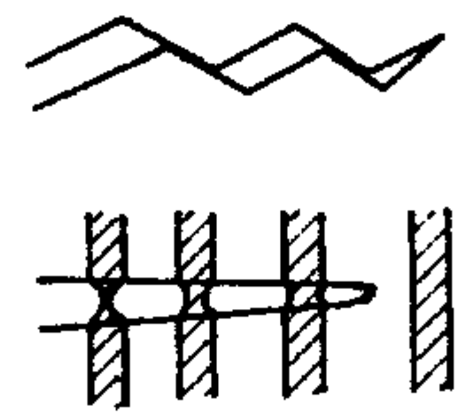
Understand toughening mechanisms:



Deflection and meandering



Shielding / micro cracks



Pinning and bridging

Ritchie, '99

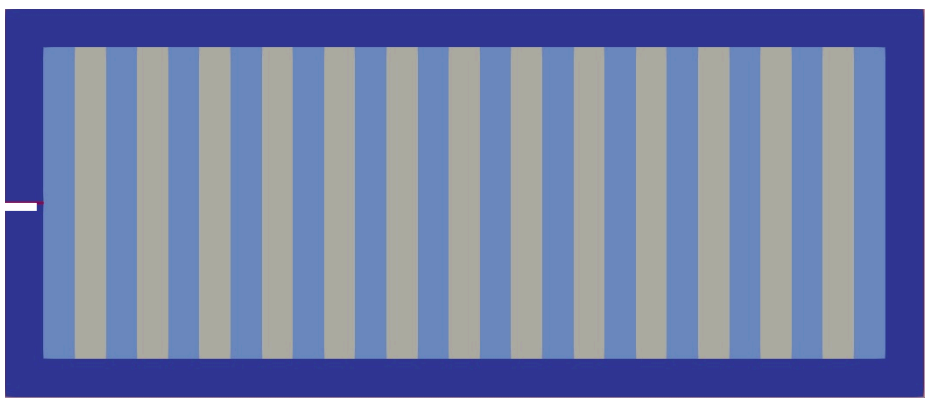
Compute "effective" fracture properties of heterogeneous materials.

Design materials with "extreme" fracture properties.

Mathematical view

Giacomini-Ponsiglione ’06, Friedrich-Perugini-Solombrino ’22, Γ-convergence of Griffith’s fracture energy (static, then quasi-static evolution).

Elastic and fracture properties homogenize separately, *toughening is impossible*

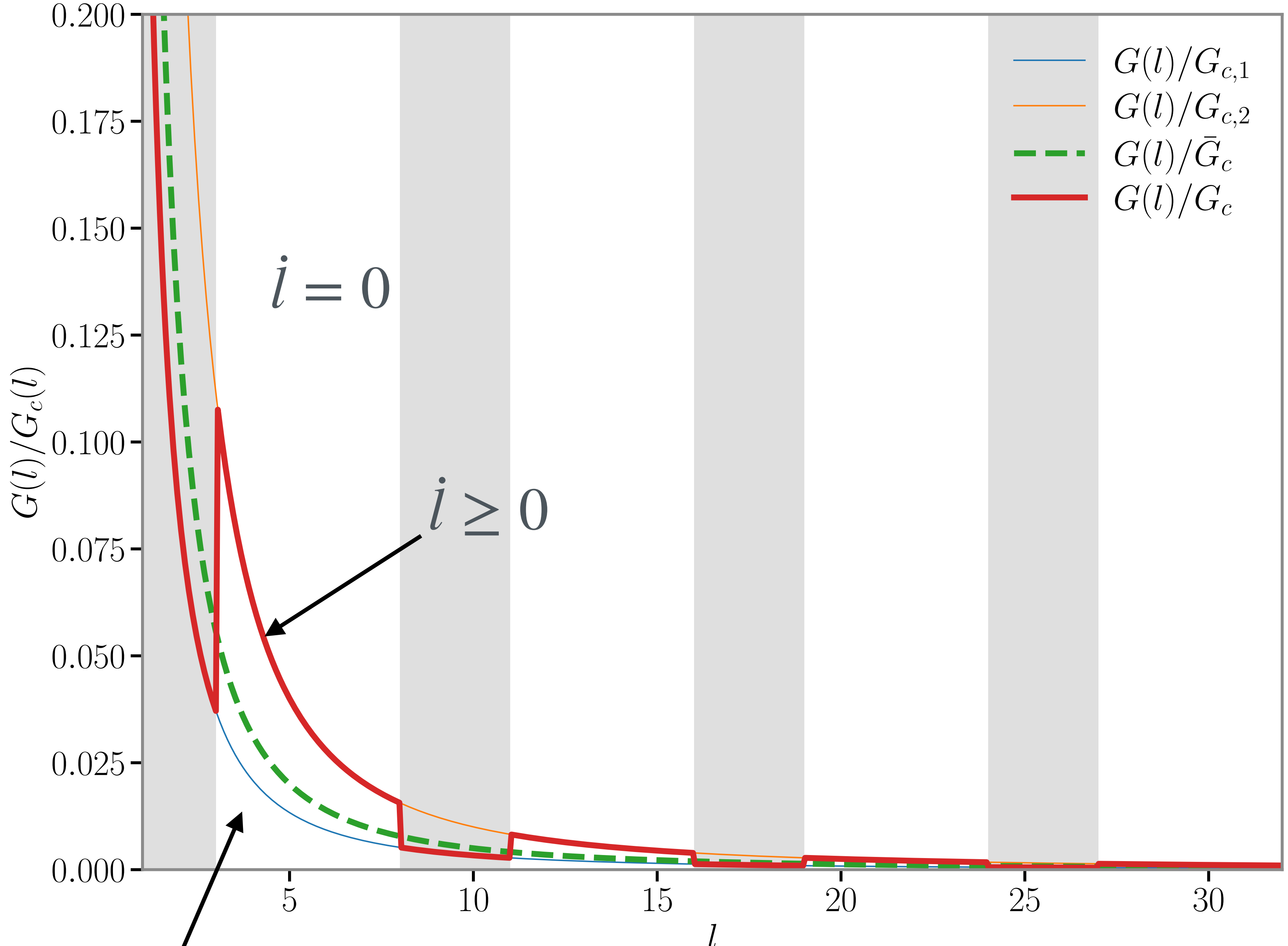


Toughness layering, M.I.L.:

$$G(t,l) = t^2 G(1,l)$$

Griffith criticality:

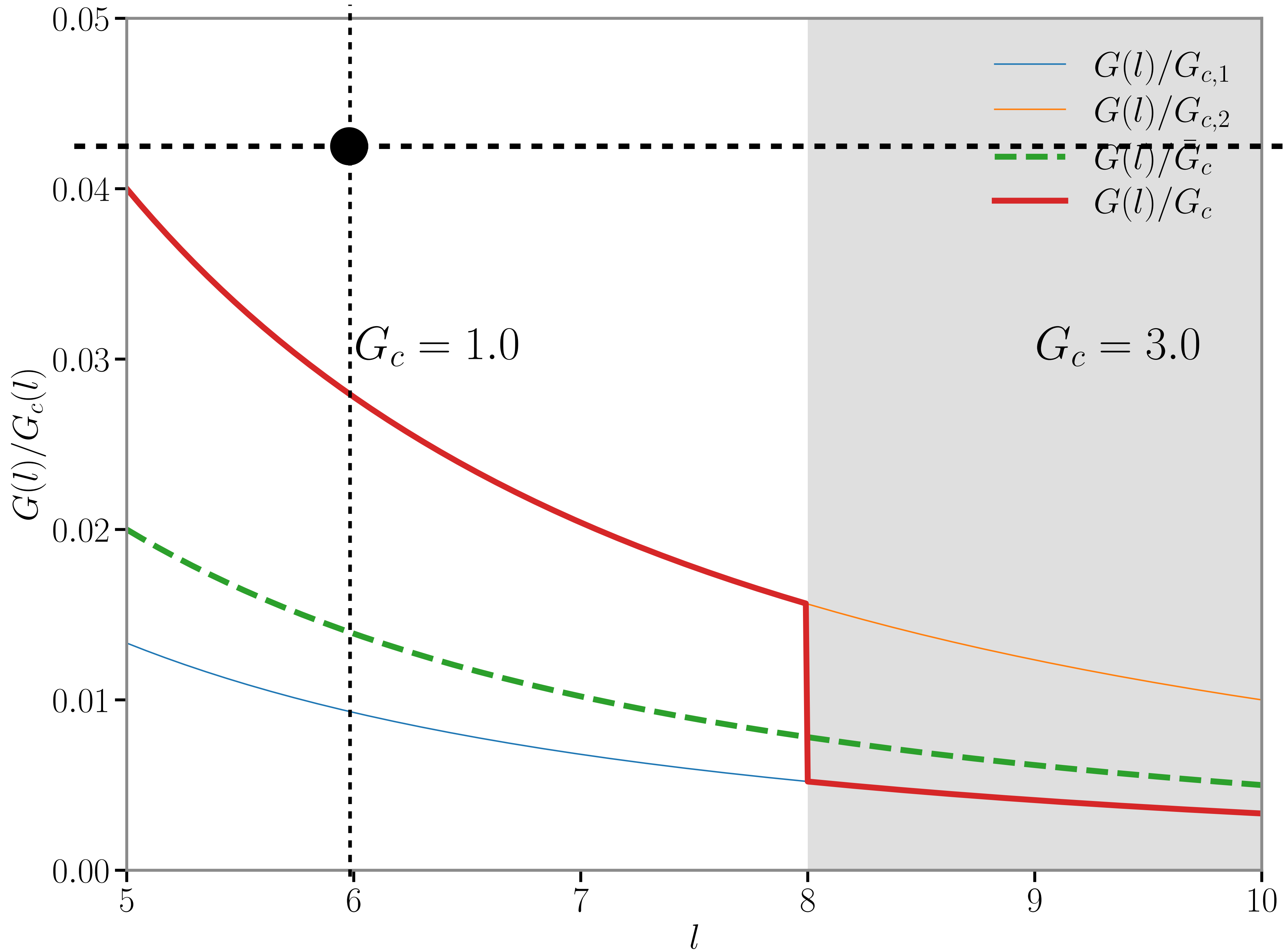
$$G(1,l)/G_c(l) = \frac{1}{t^2}$$



Forbidden!

Weak to tough transition

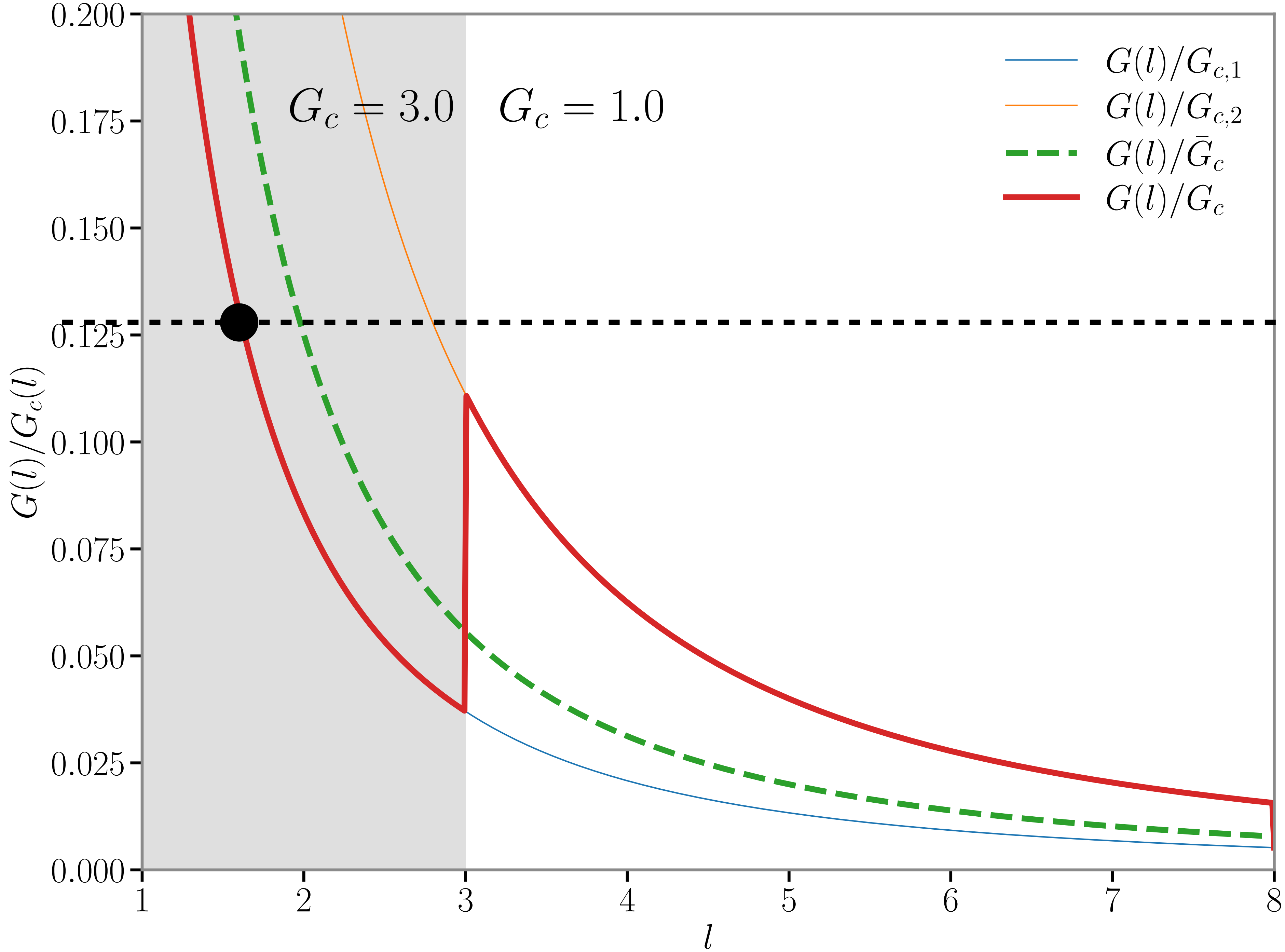
Evolution is unambiguous



Griffith criticality:

$$G(1,l)/G_c(l) = \frac{1}{t^2}$$

Tough to weak transition

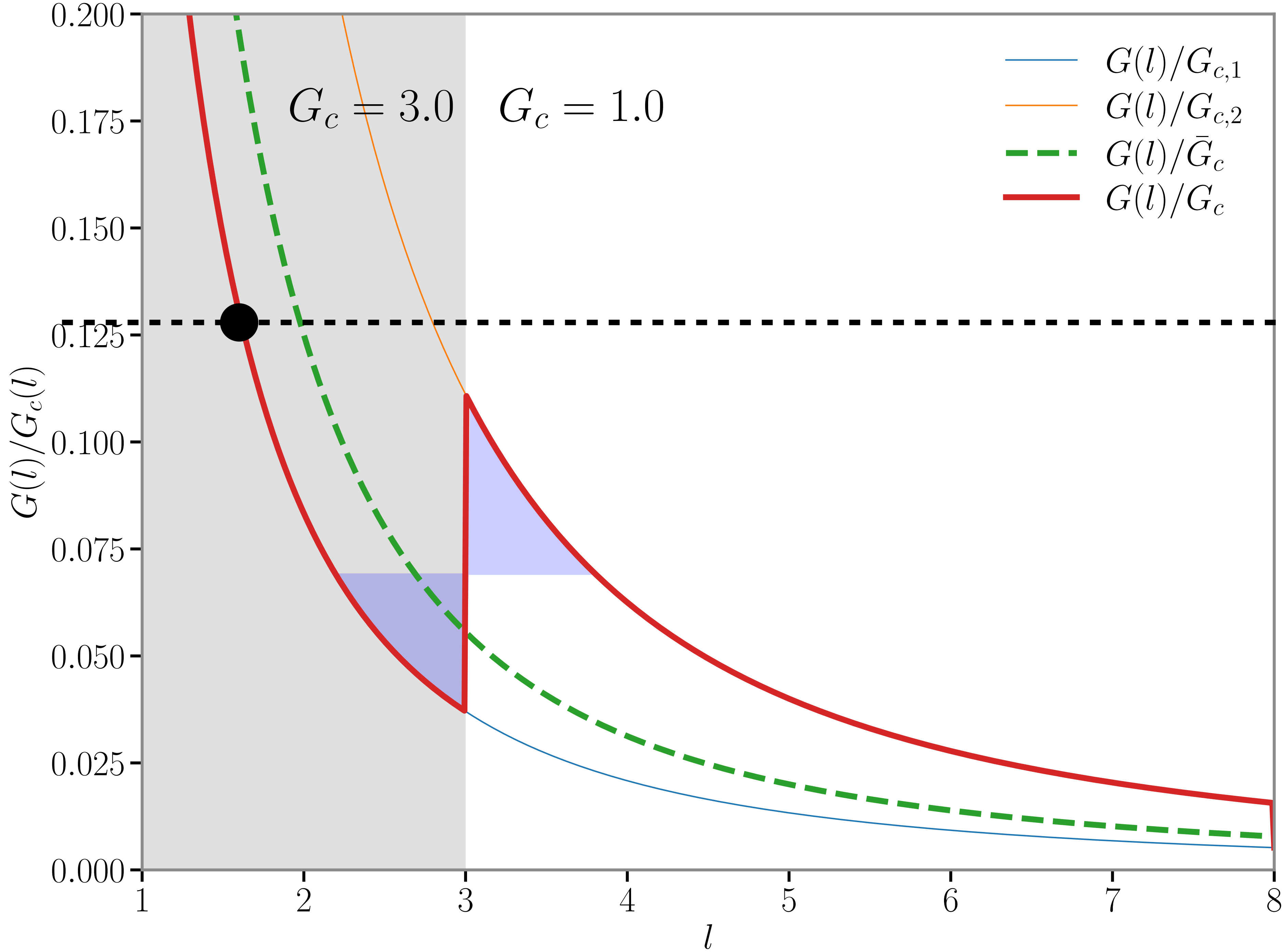


Griffith criticality:

$$G(1,l)/G_c(l) = \frac{1}{t^2}$$

Tough to weak transition

Global minimality breaks causality



Griffith criticality:

$$G(1,l)/G_c(l) = \frac{1}{t^2}$$

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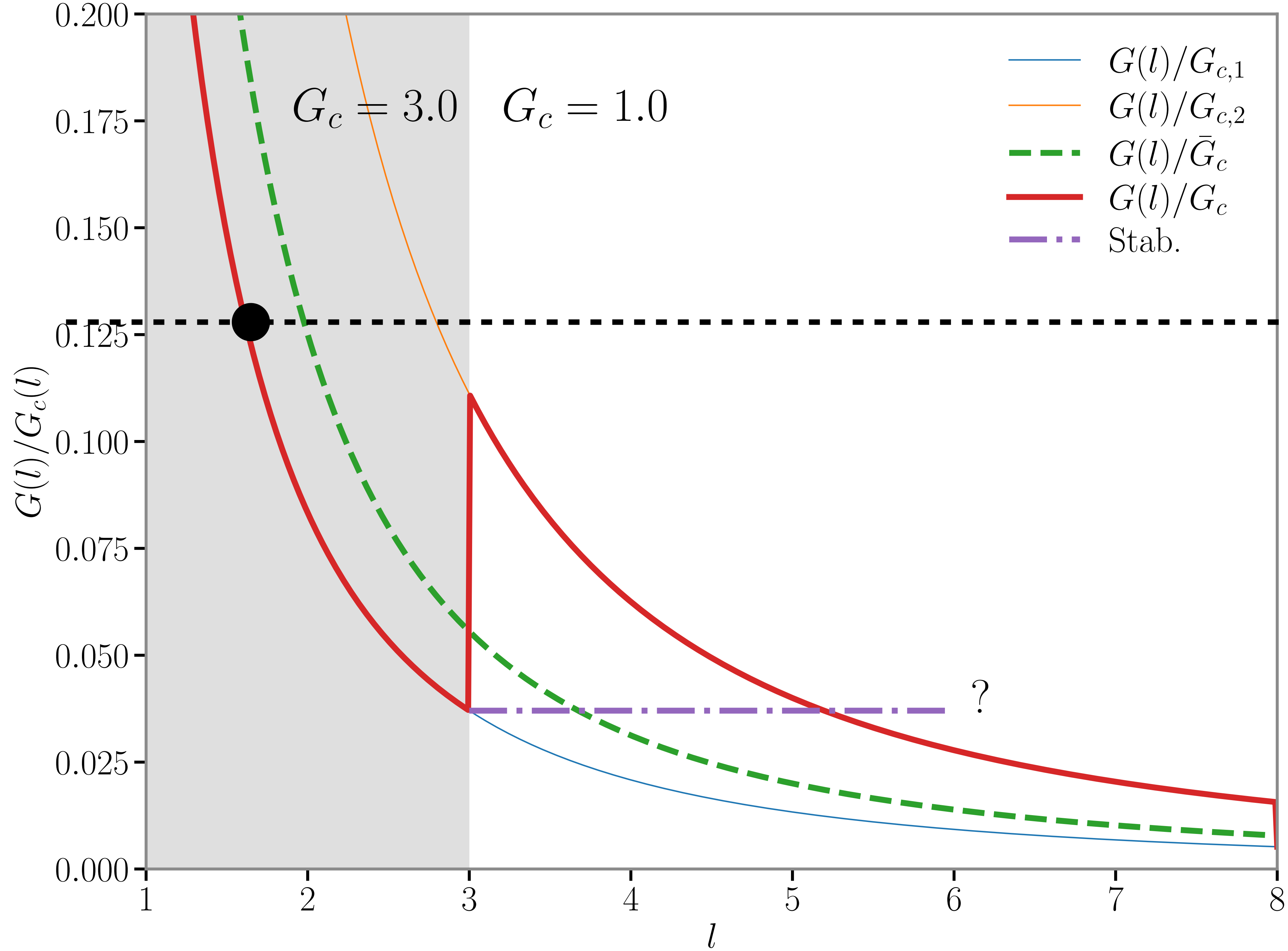
Tough to weak transition

Stability + energy balance

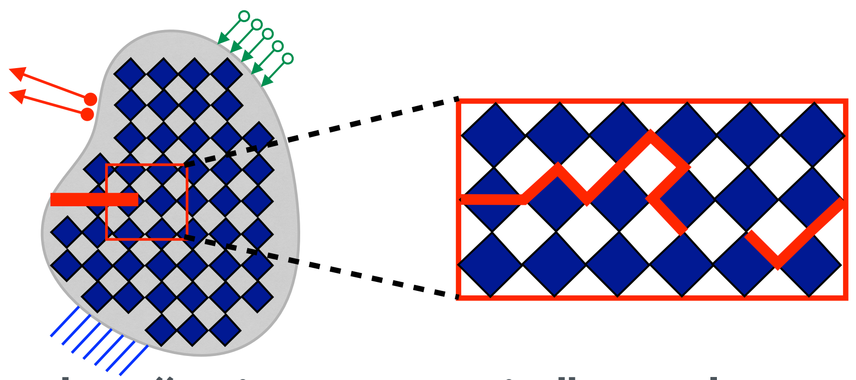


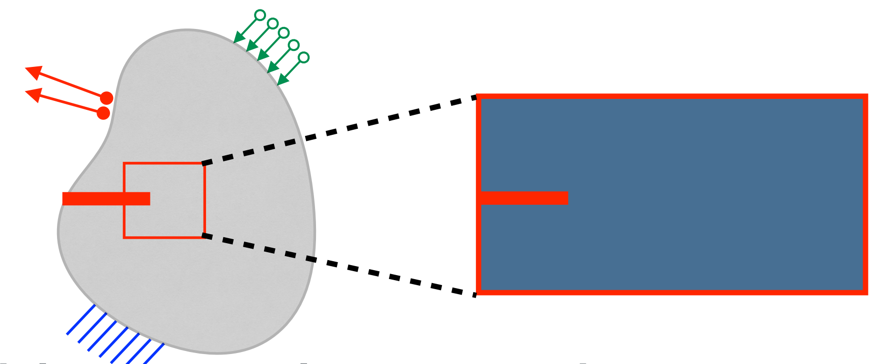
Griffith criticality:

$$G(1, l)/G_c(l) = \frac{1}{t^2}$$

An empirical concept of effective toughness

Problem: Micro-geometry defined by $\mathbf{A}^\varepsilon$, G_c^ε, define G_c^{eff} such that $G_c^\varepsilon \to G_c^{\mathrm{eff}}$ while accounting for causality, energy barriers, etc.
"homogenization in trajectory space".



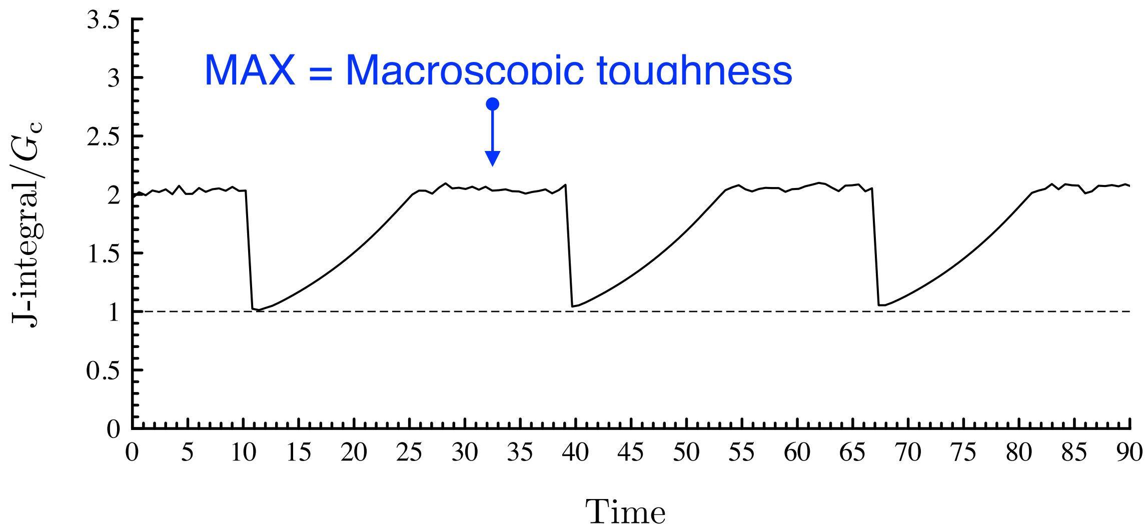


At the "microscopic" scale, evolution by stable critical points, discontinuous evolution, no energy balance: energy barriers.

At the macroscopic scale, periodic elastic energy release rate.

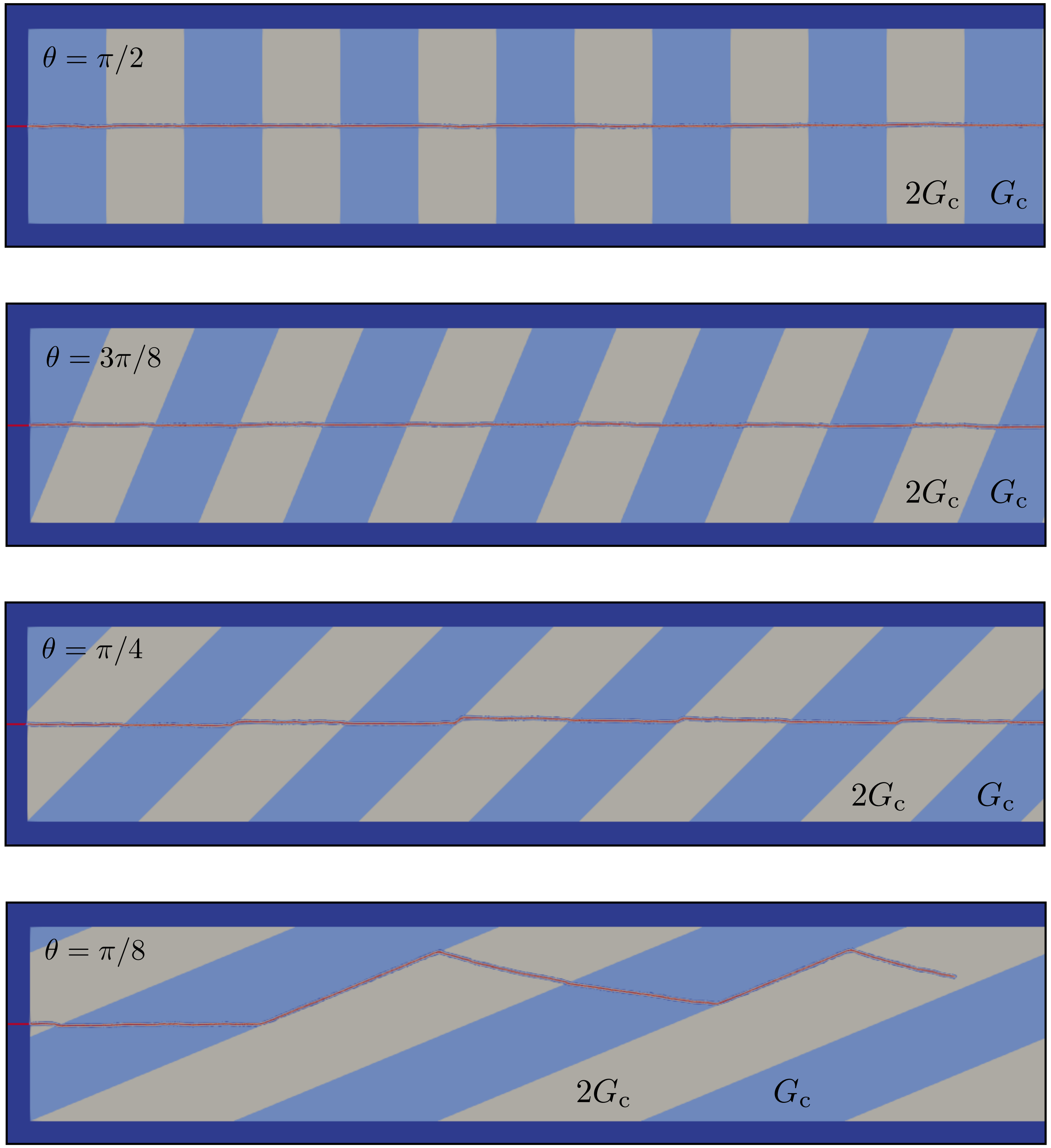
Proposed concept of effective toughness:

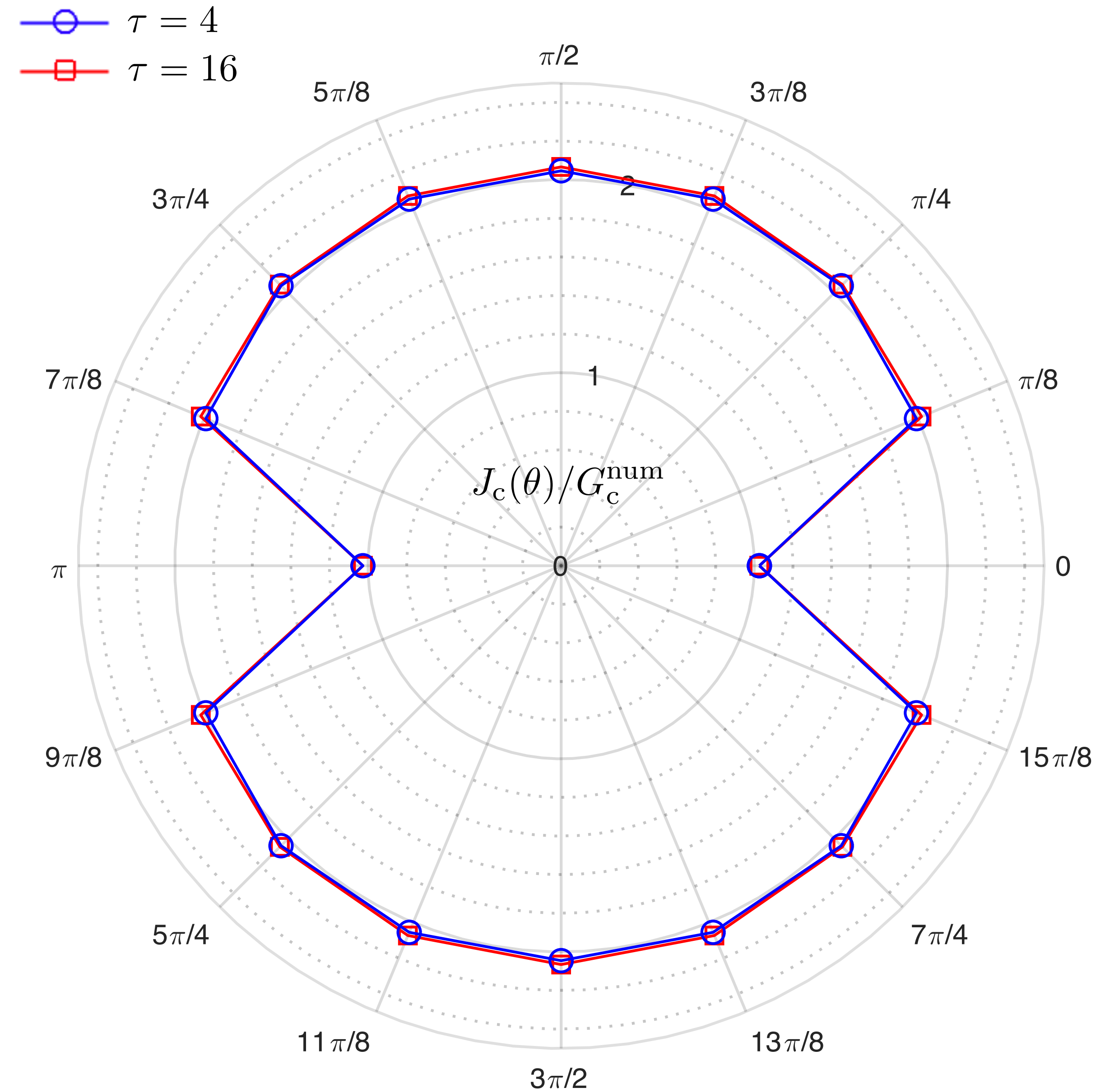
$$G_c^{\mathrm{eff}} = \lim_{\varepsilon \to 0} \sup_{k\varepsilon \leq l \leq (k+1)\varepsilon} G(l)$$



Hossain et al, *JMPS*, 2014.

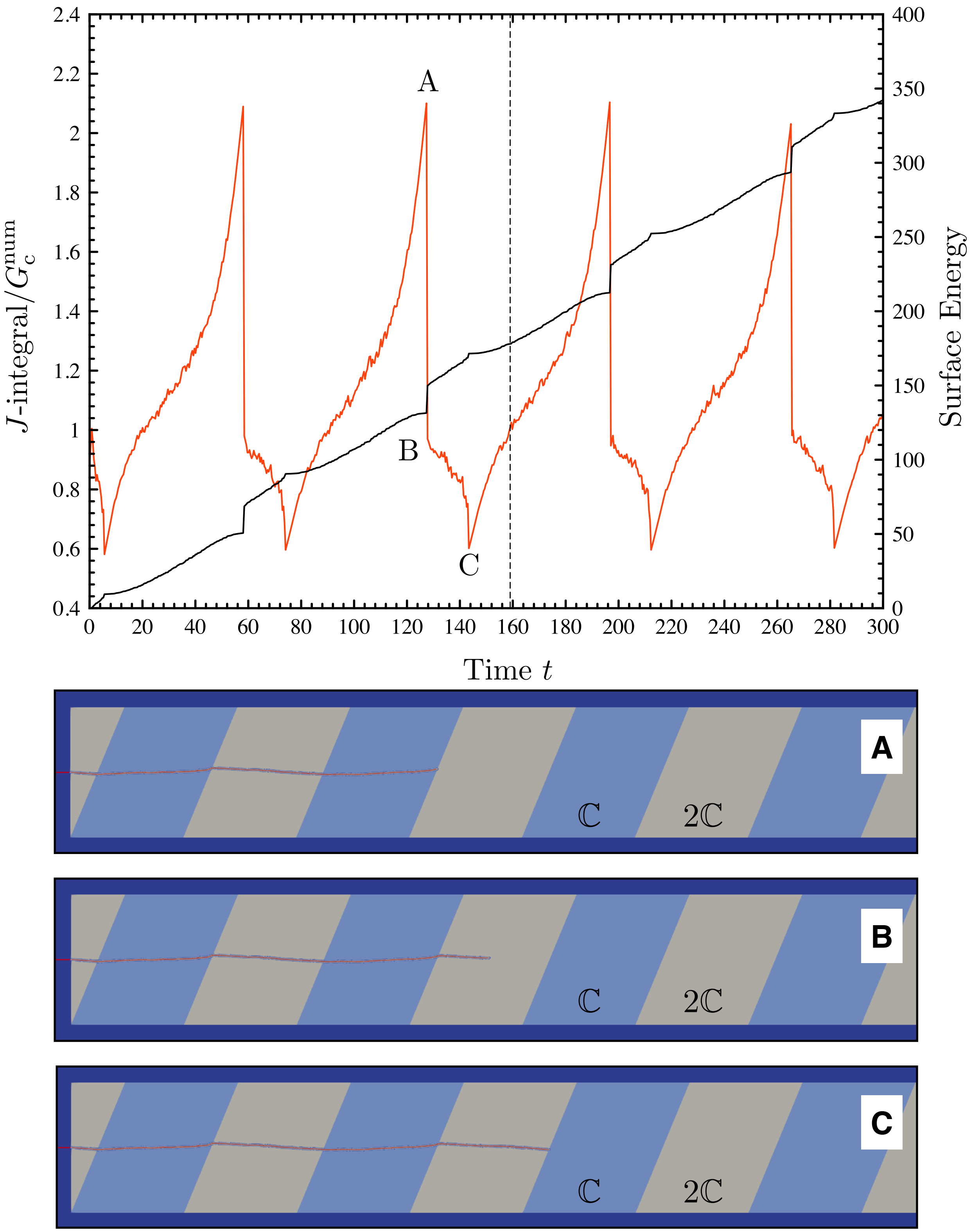
Toughness heterogeneities

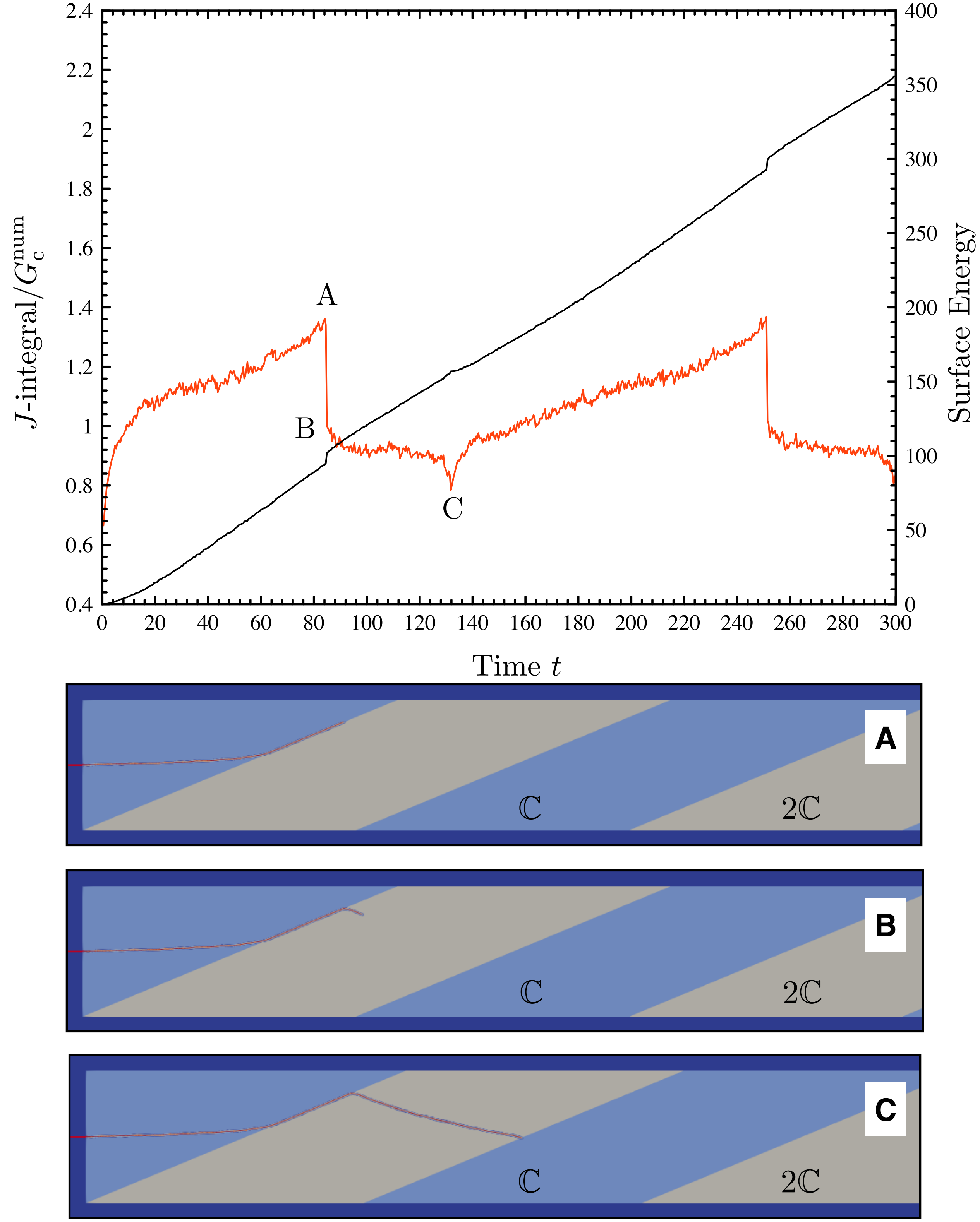




Brach, Hossain, B, Bhattacharya *JMPS* '19

Elastic heterogeneities





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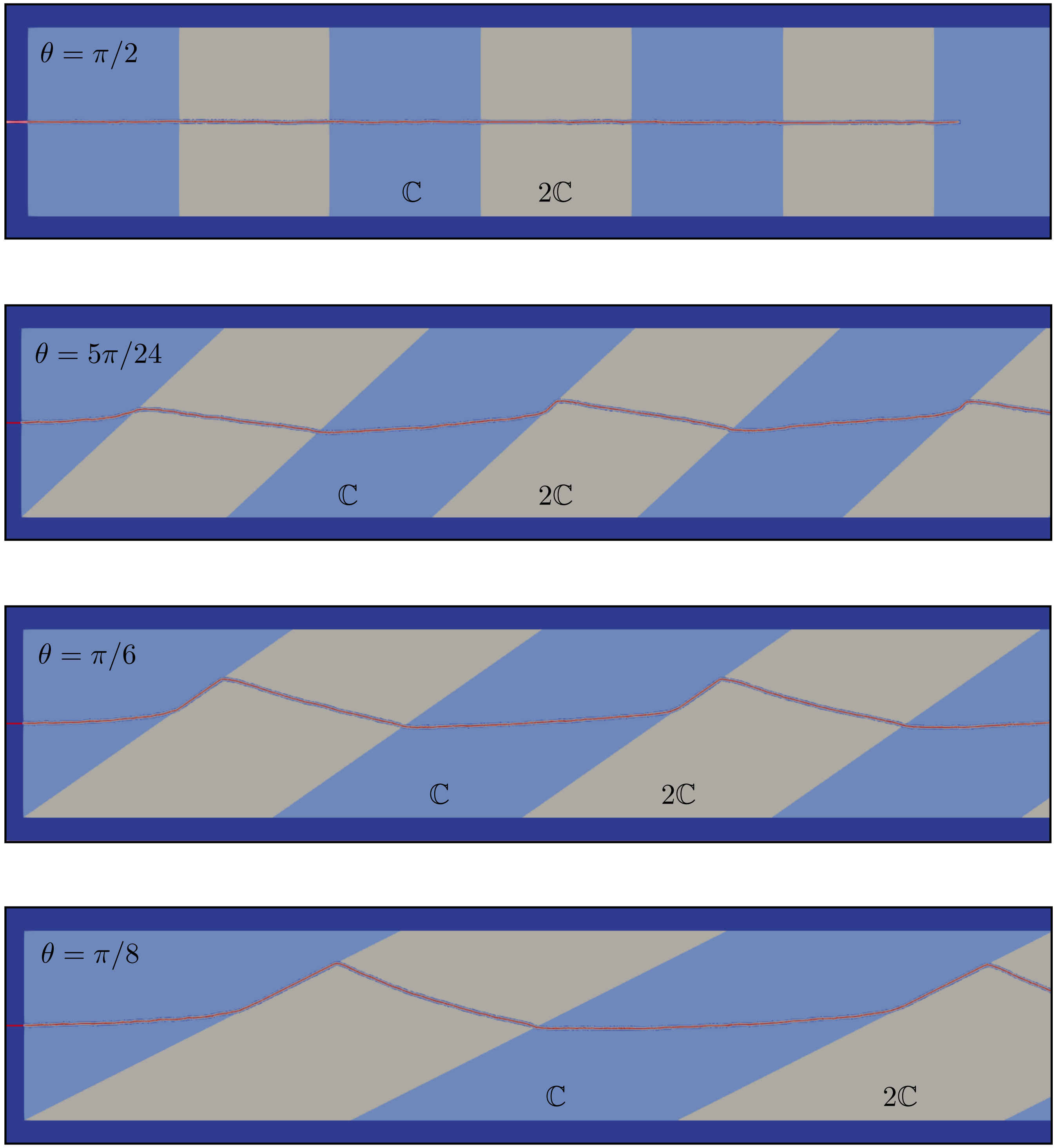
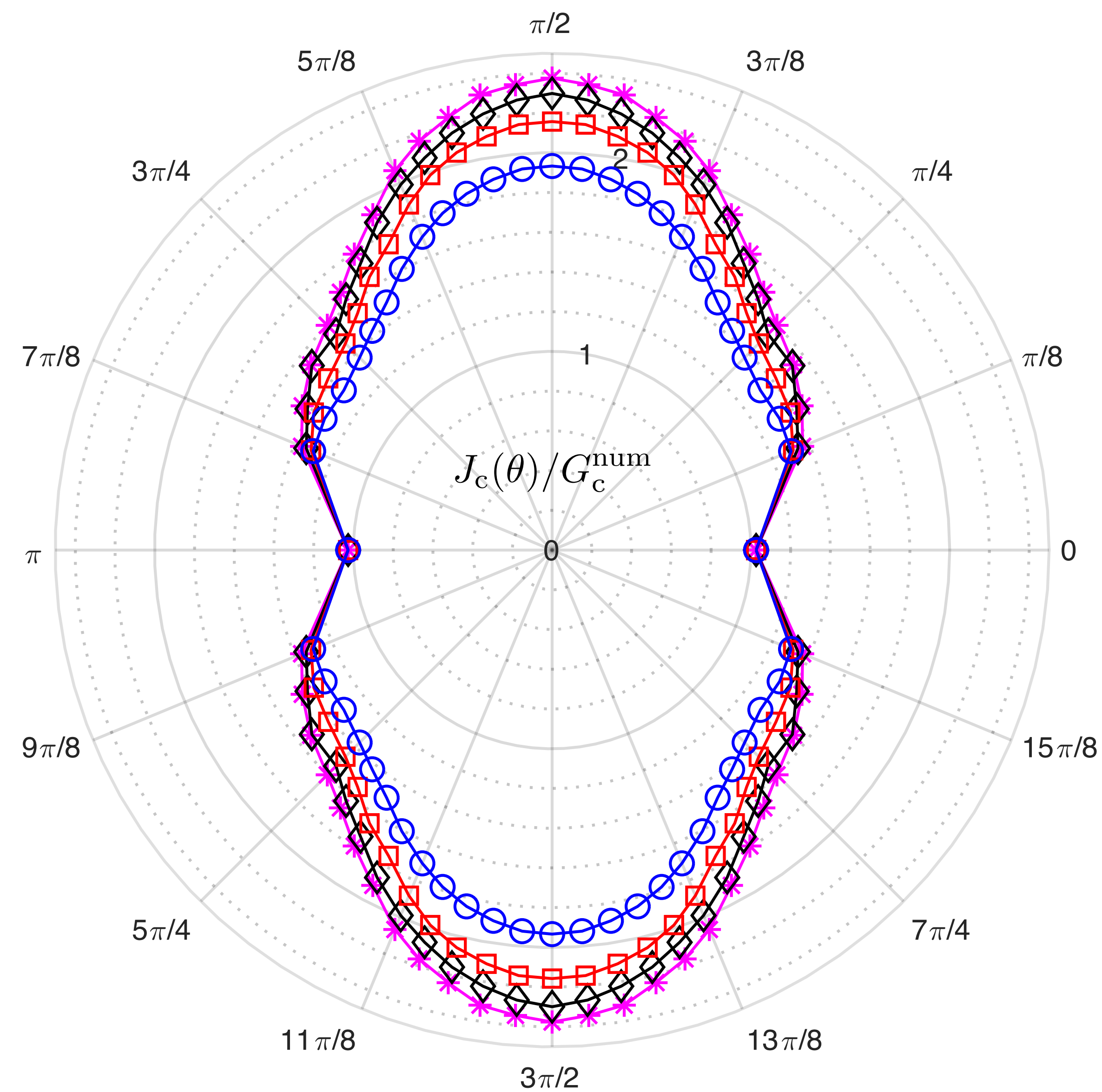
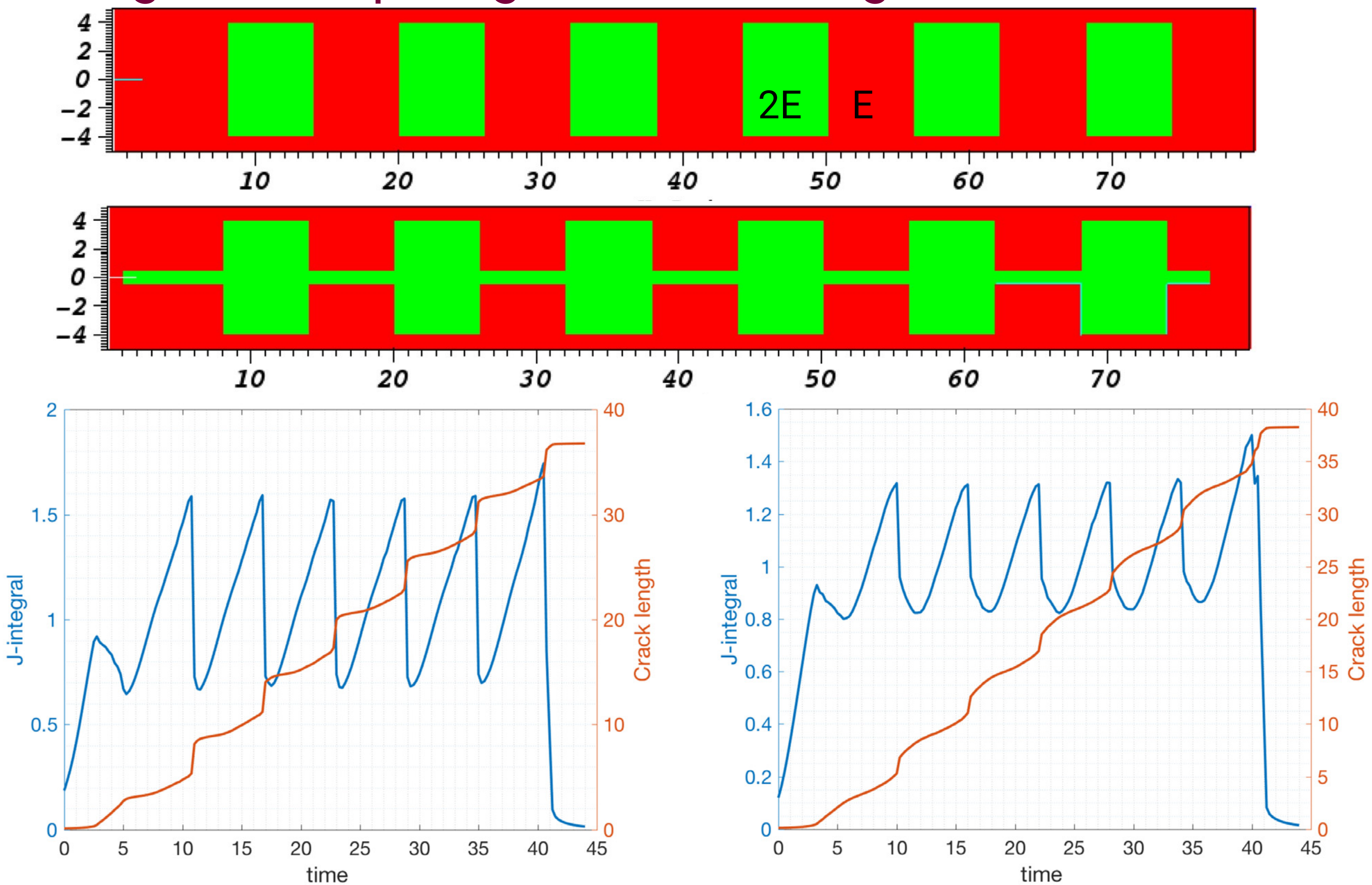
Elastic heterogeneities





Toughening without pining or meandering

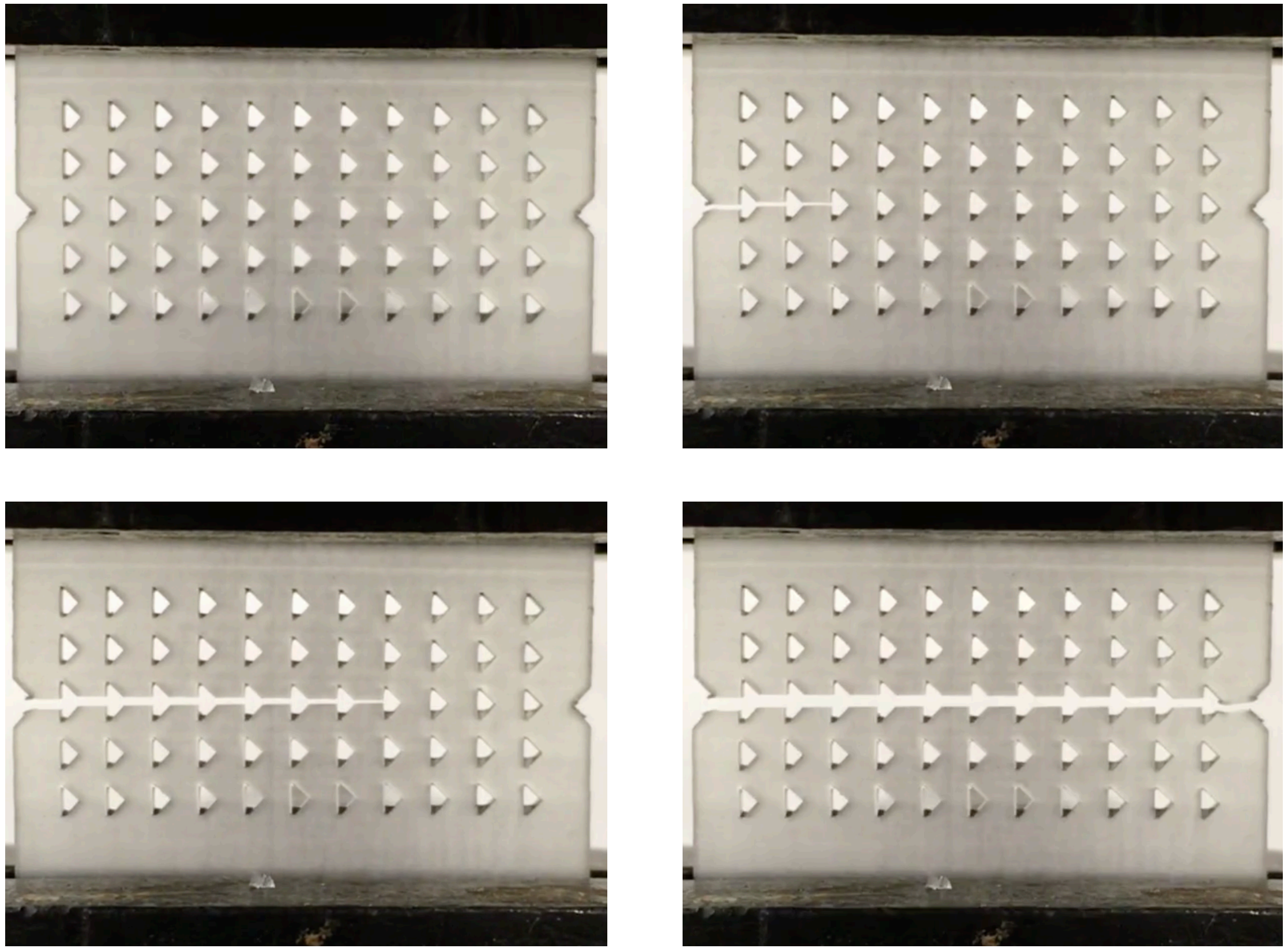


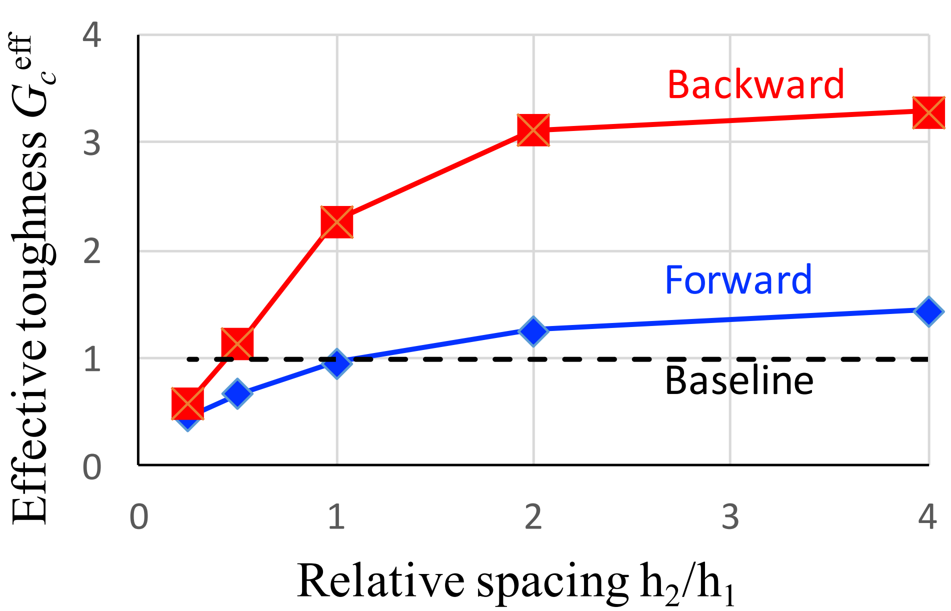
Layers

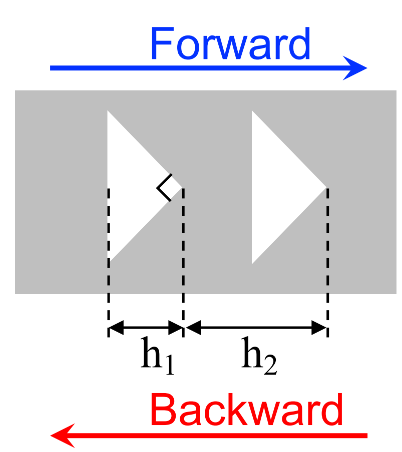
Channels

Hsueh, Avellar, B, Ravichandran, Bhattacharya, *JMPS* '18

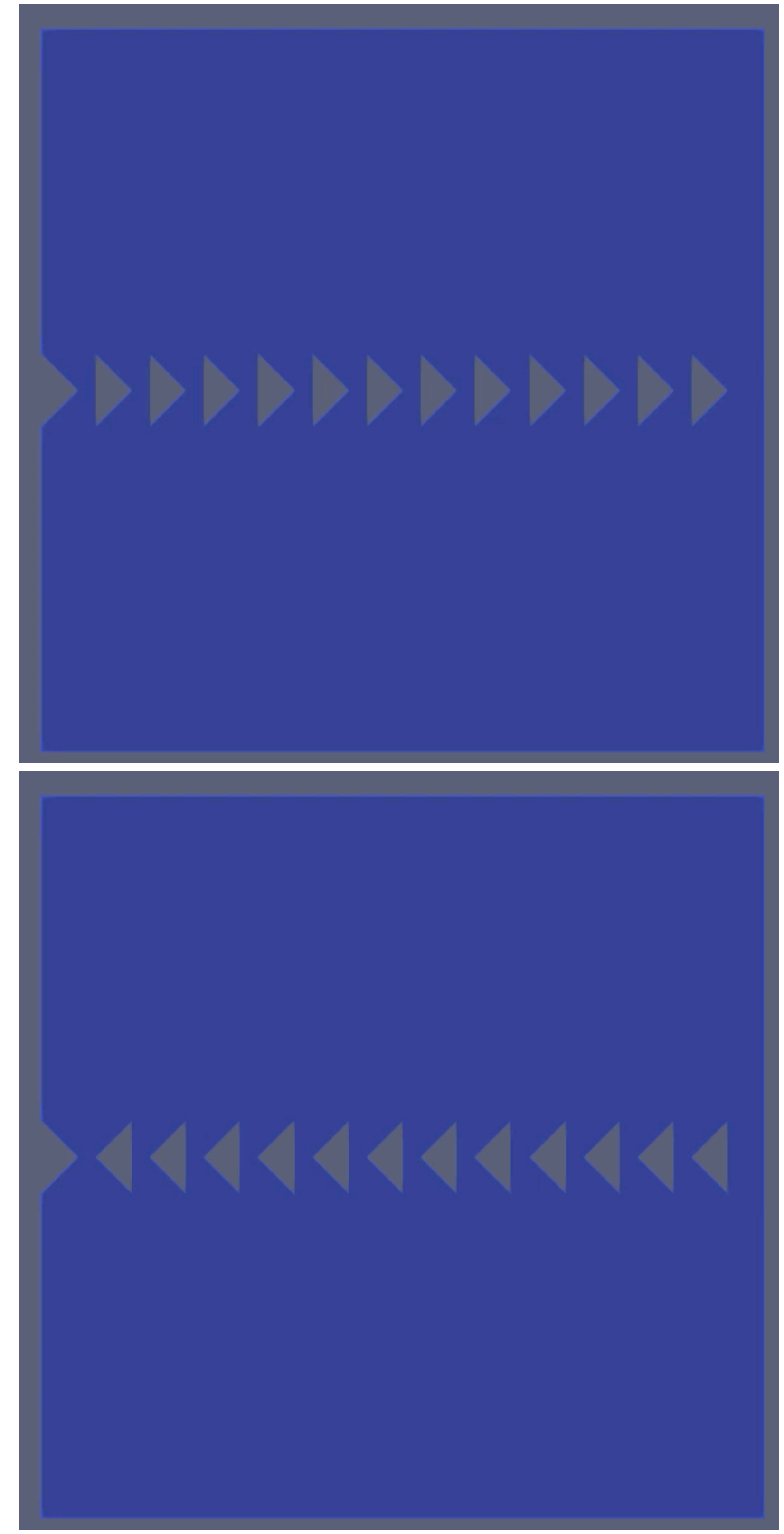
Fracture diodes: directionally anisotropic toughness







Brodnik et al, PRL '21



Conclusions

Phase-field models have demonstrated their ability to handle crack propagation in a broad range of materials, loading (including complex multi-physics settings).

Numerical evidence that mode-I nucleation in compressible materials can be accounted for.

Open problems:

- Stress (not elastic energy density) nucleation criterion.
- Can nucleation be fully accounted for in a variational setting?
- Can nucleation and Griffith-like energies be reconciled?
    - Cohesive fracture? Ductile fracture? Dynamic fracture?
- Mathematical framework for evolution of meta-stable states. Alternative to Γ-convergence to connect phase-field models and fracture.
- Rigorous concept of effective toughness.

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